Decision Diagrams for Integer Linear and Nonlinear Programming

Willem-Jan van Hoeve (Carnegie Mellon University)

Joint work with:

Danial Davarnia (Iowa State University)

Christian Tjandraatmadja (Google)

EURO, June 2019

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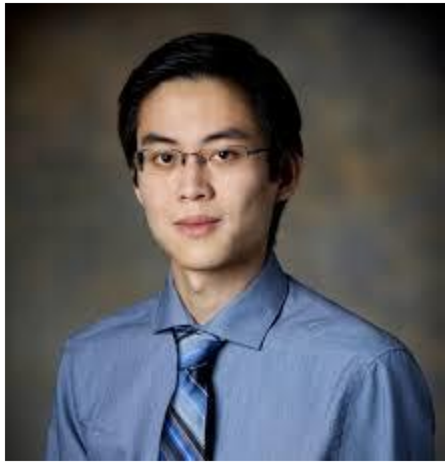

EURO, June 2019

Overview

- Motivation
- Decision Diagrams for Integer Programming
  - incorporate DD bounds in MIP search
  - cut generation
  - outer approximation for MINLP
- Conclusions

Decision Diagrams and Integer Feasible Sets

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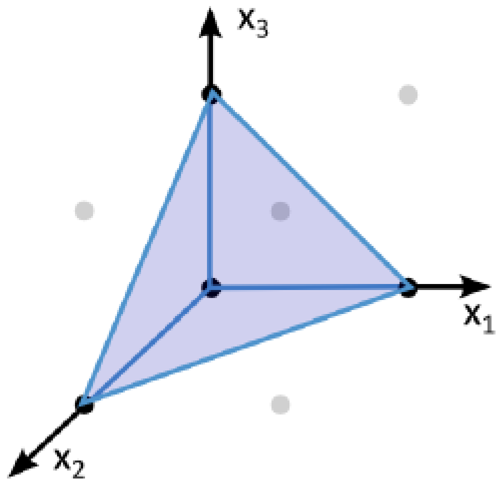

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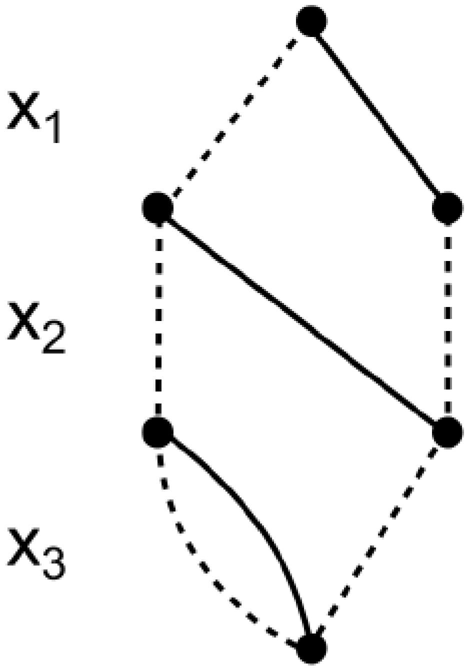

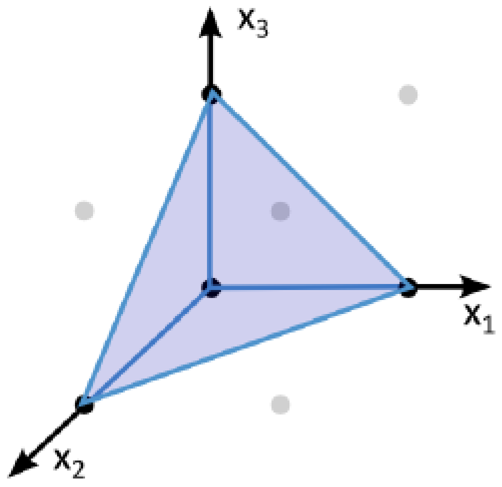

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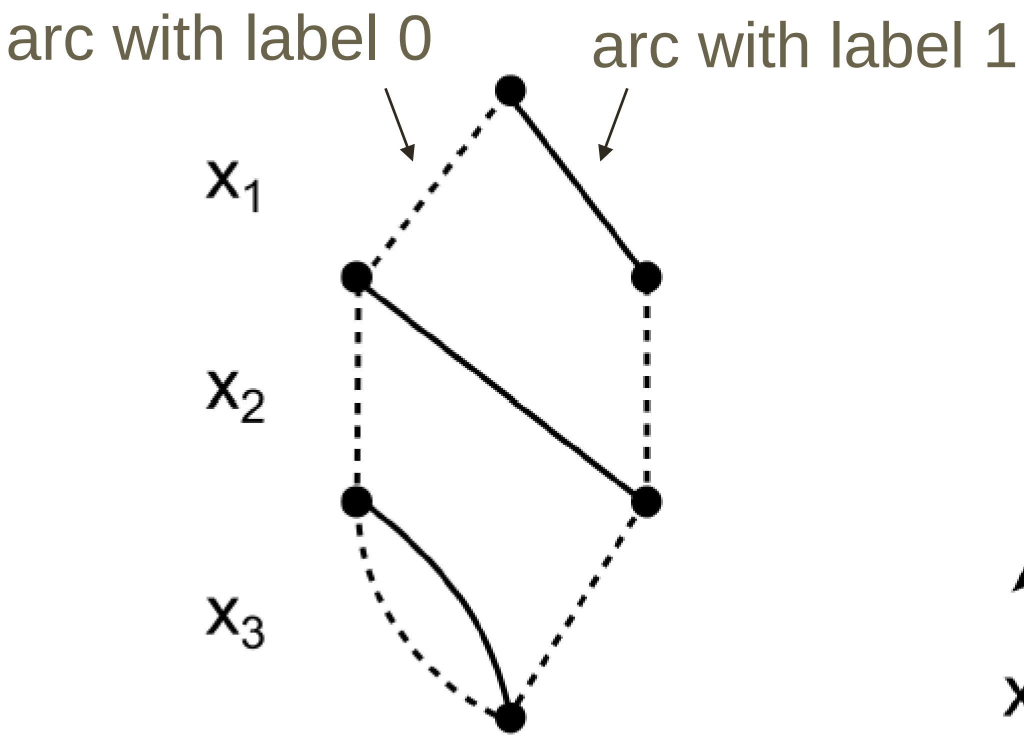

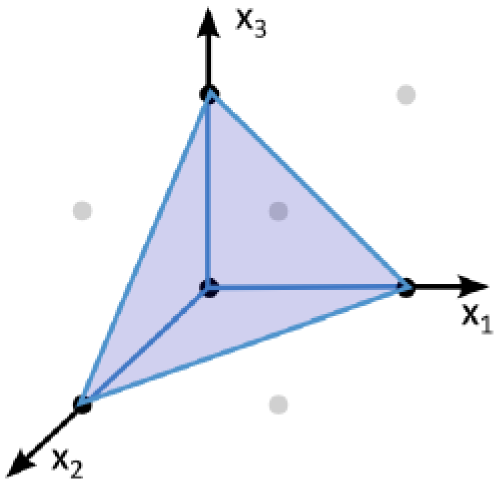

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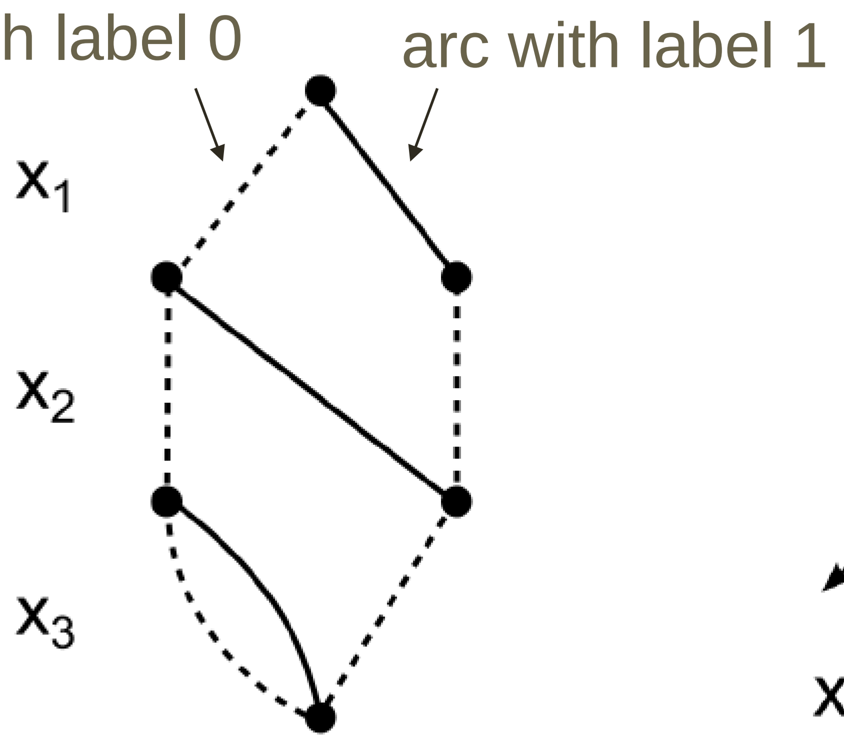

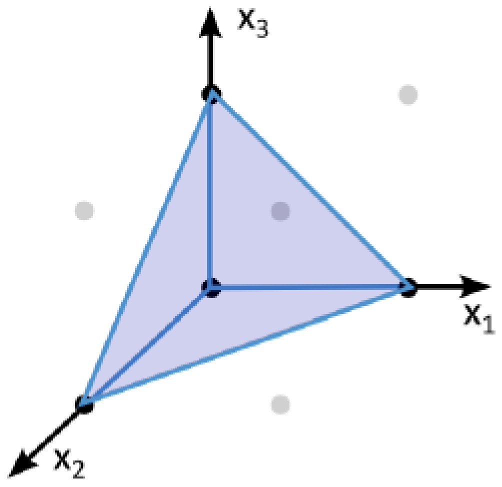

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BDD: binary decision diagram
MDD: multi-valued decision diagram

Optimizing Over Decision Diagrams

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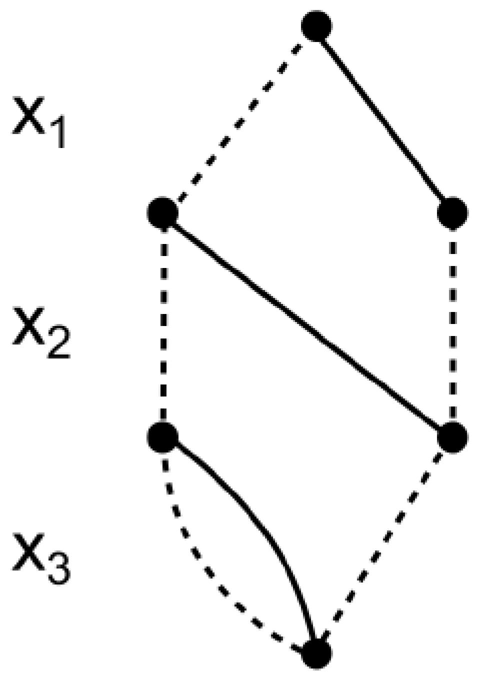

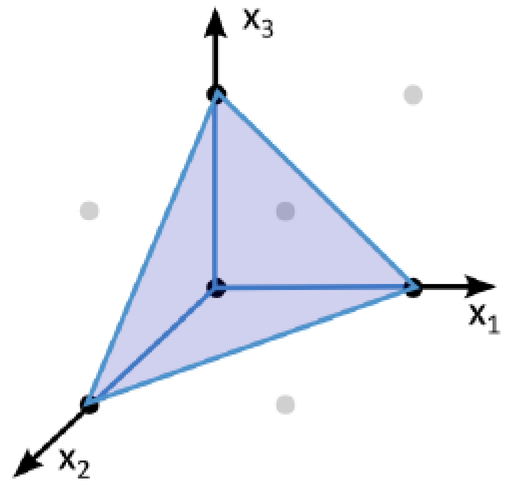

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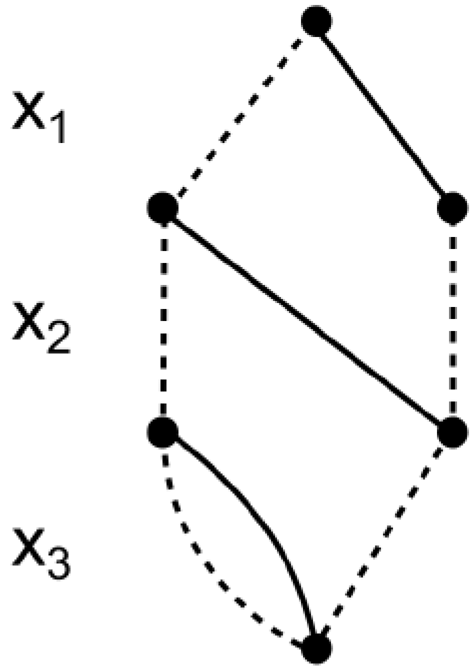

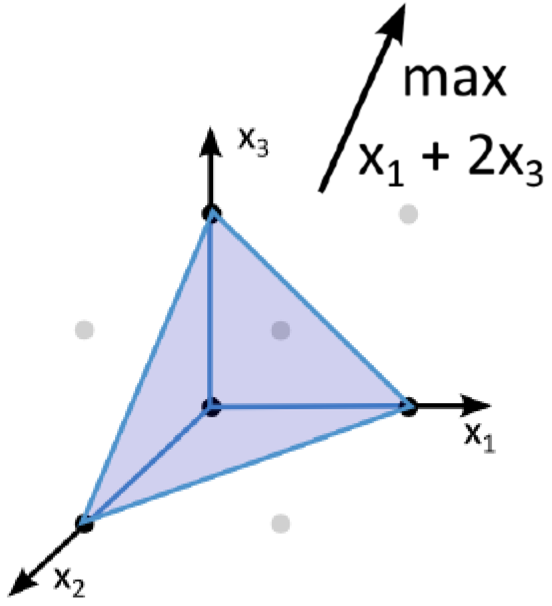

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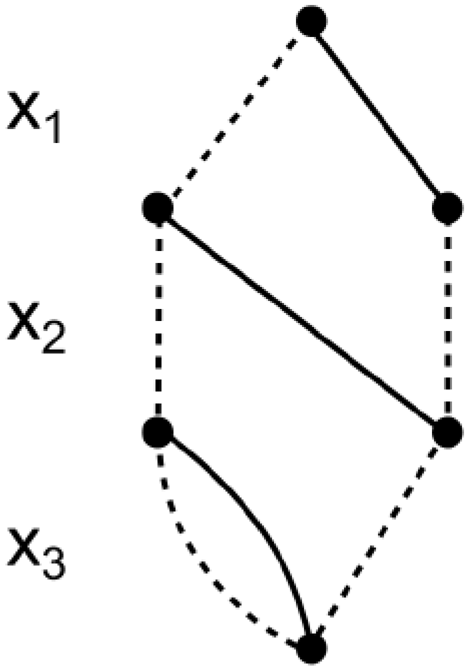

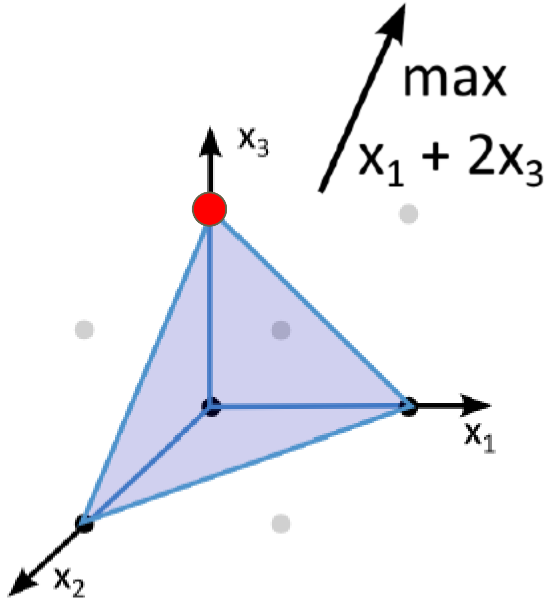

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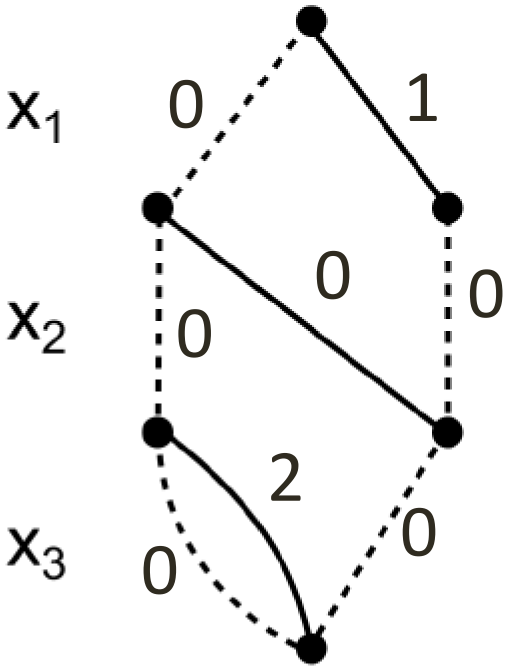

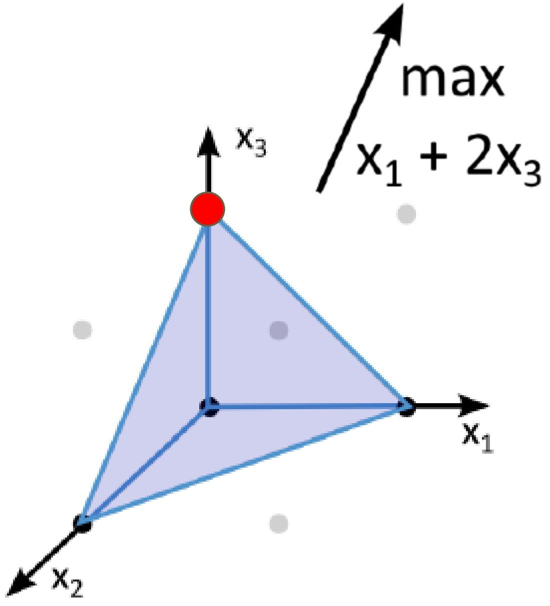

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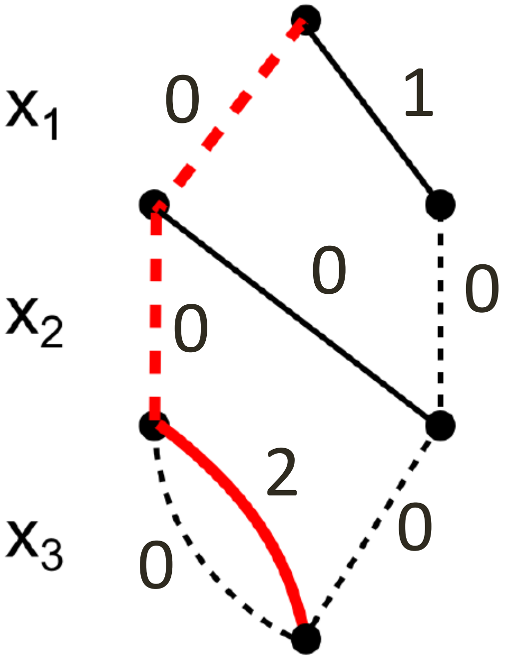

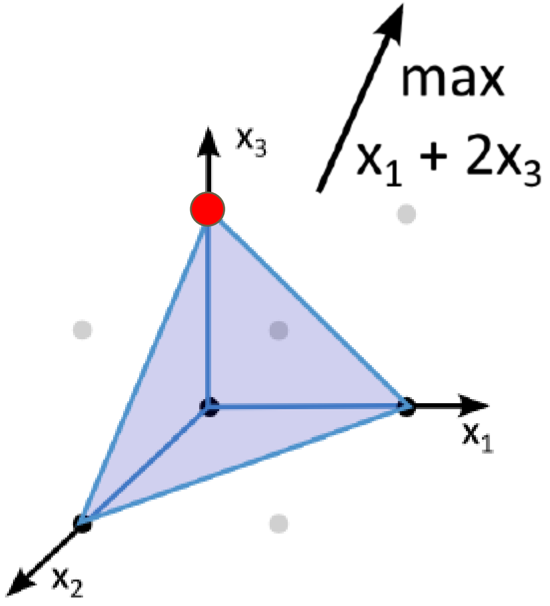

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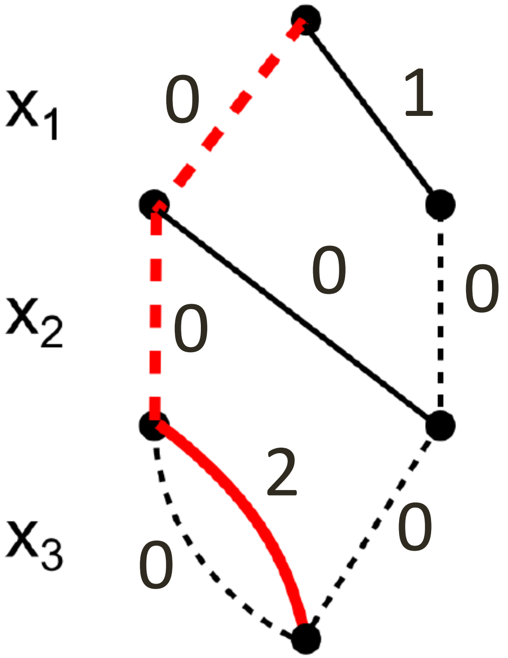

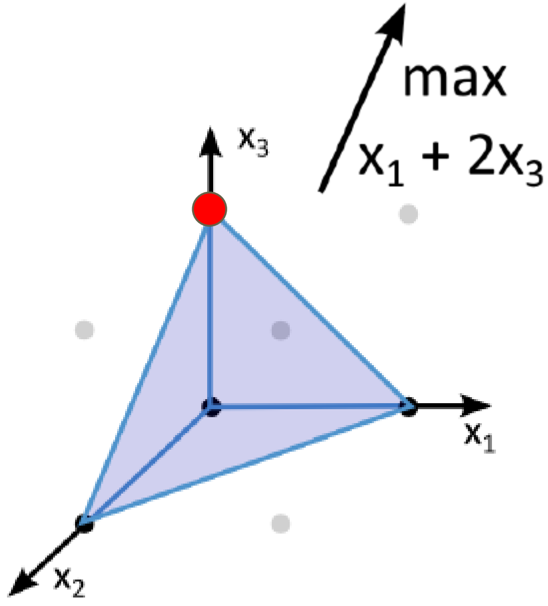

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optimal objective value: 2

Relaxed Decision Diagrams

- Relaxed Decision Diagrams have limited width: polynomial size

[Andersen et al. 2007]

- Over-approximation of feasible set: dual bound

[Bergman et al. 2011, 2014]

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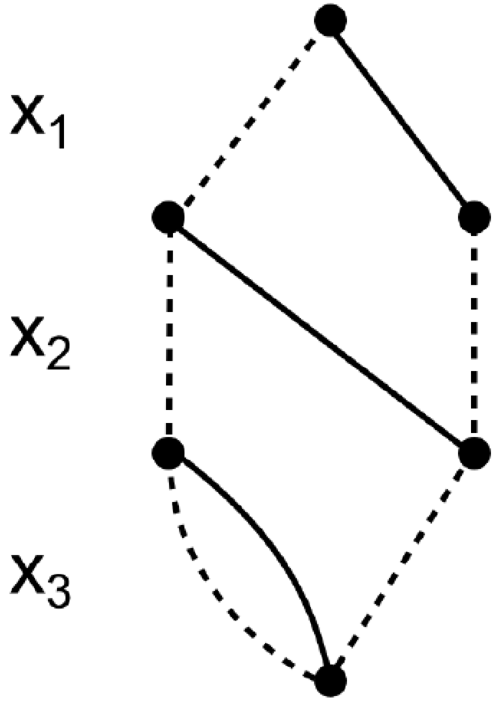

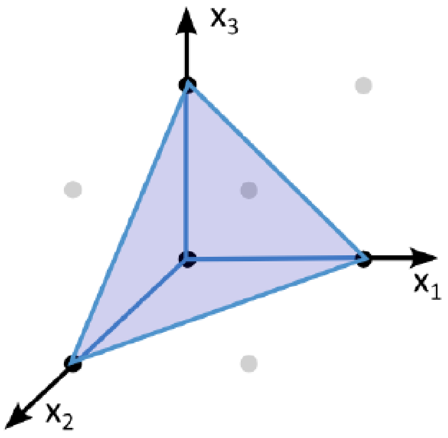

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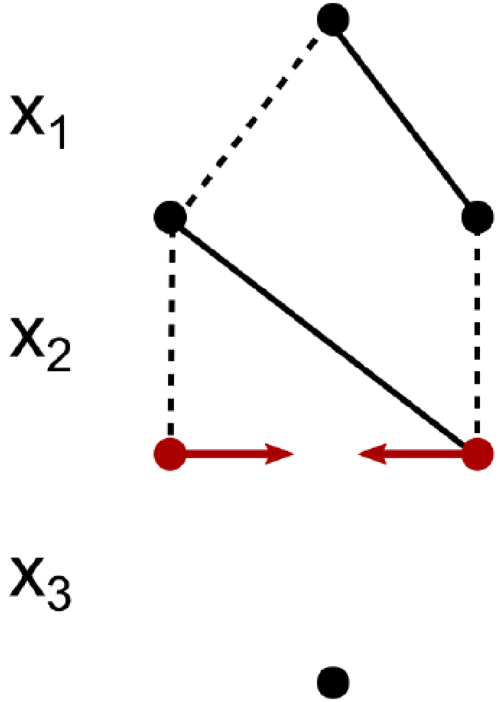

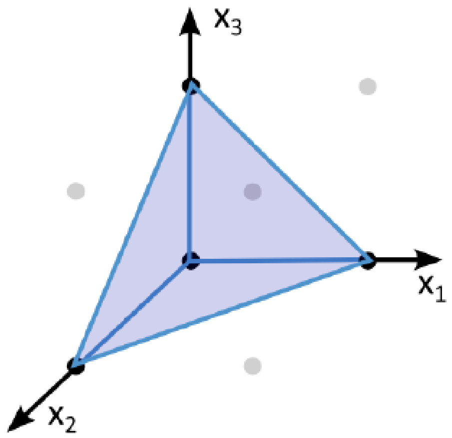

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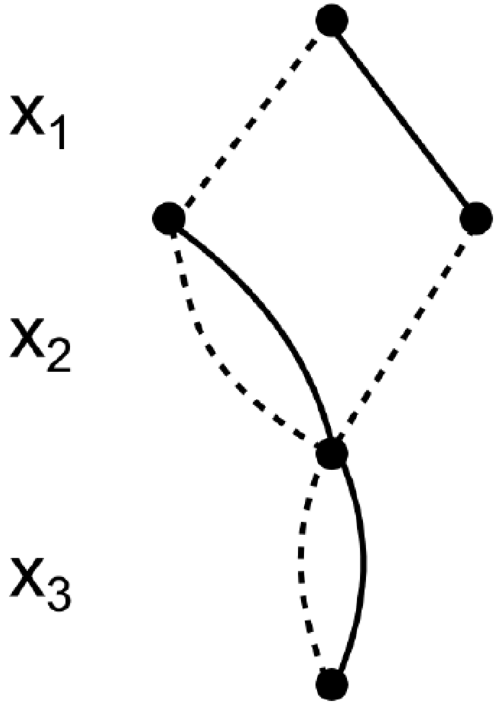

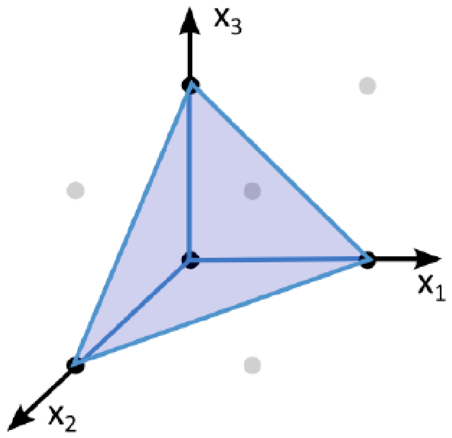

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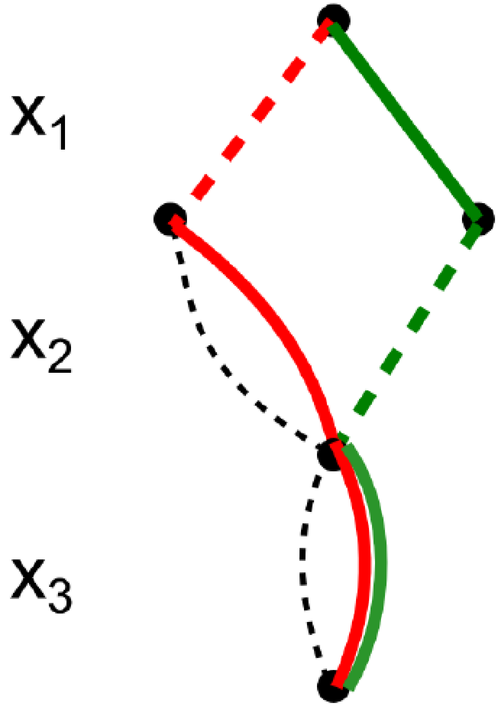

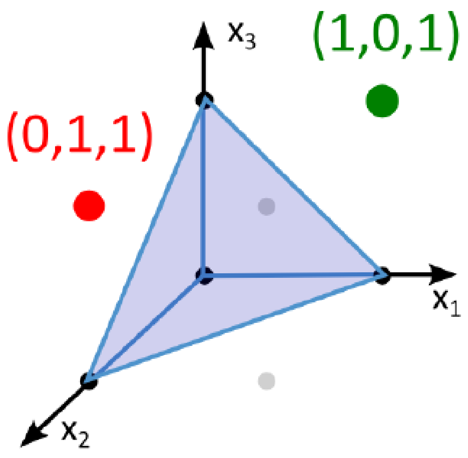

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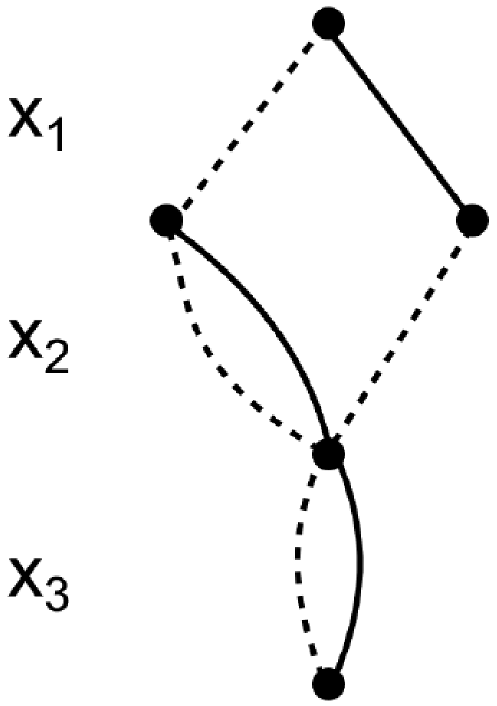

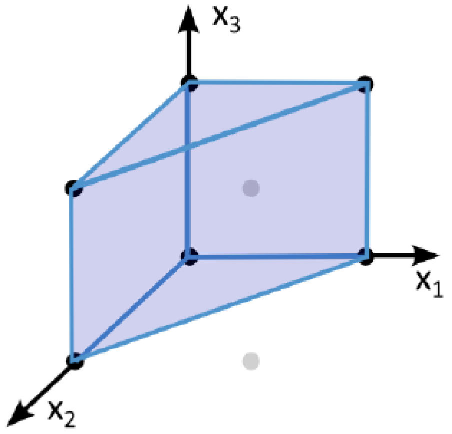

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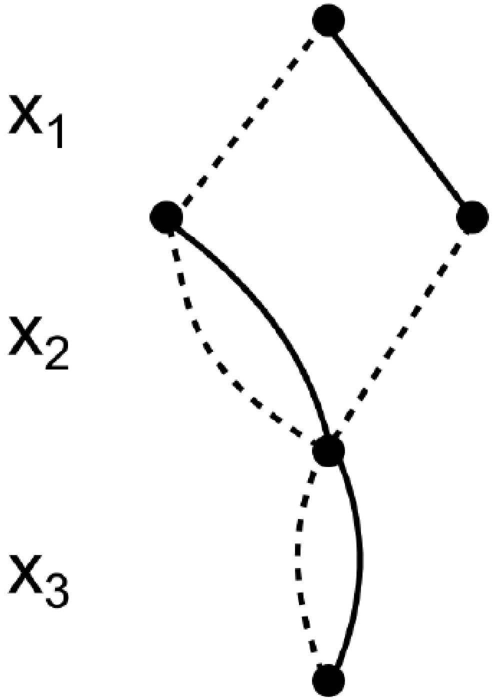

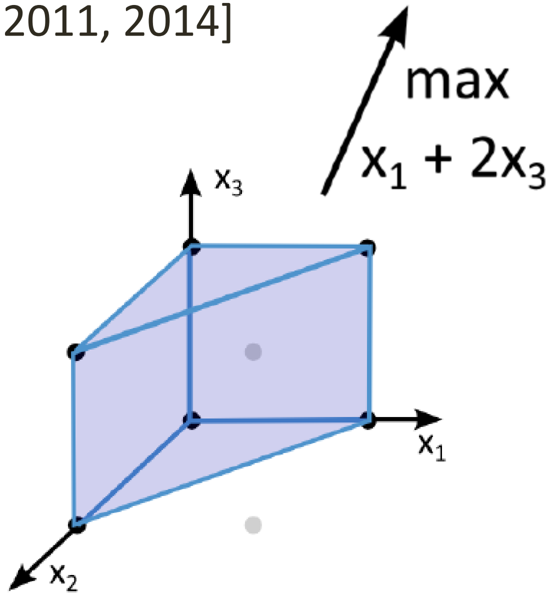

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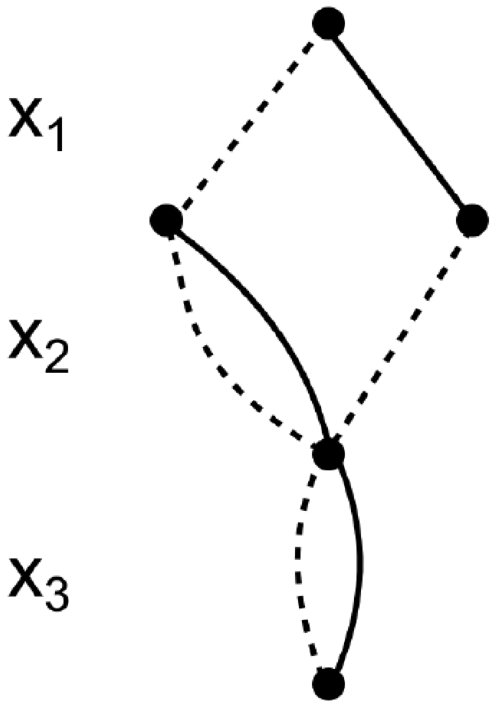

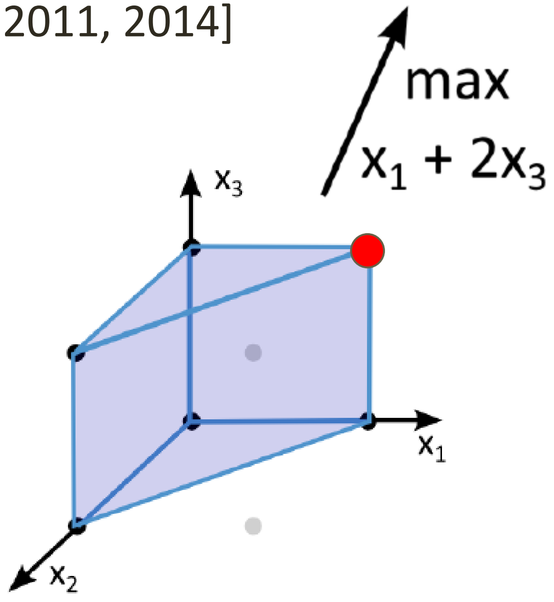

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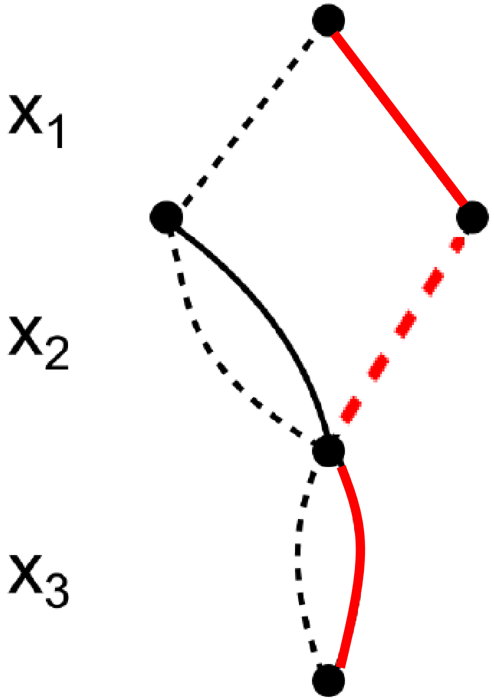

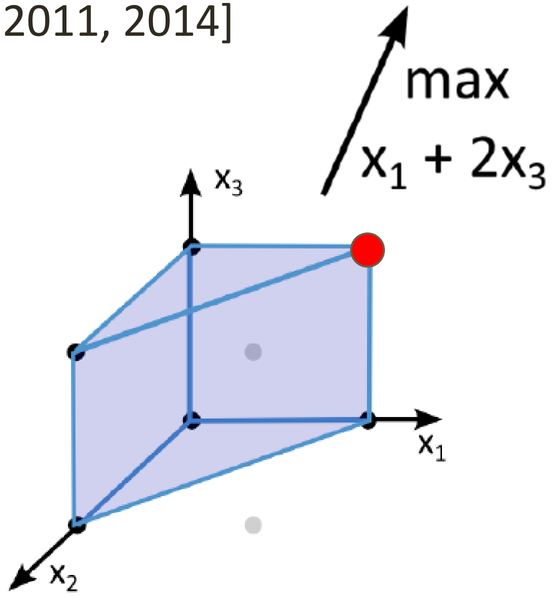

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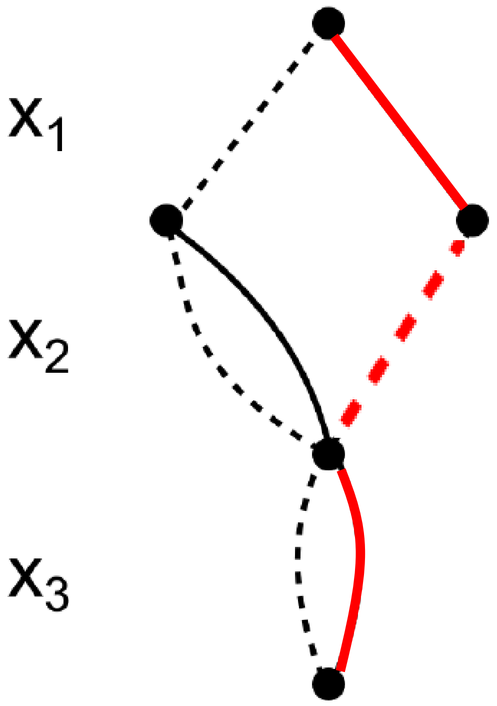
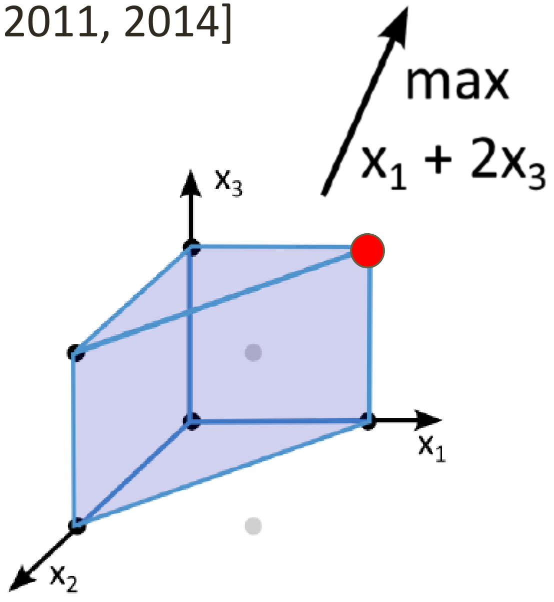


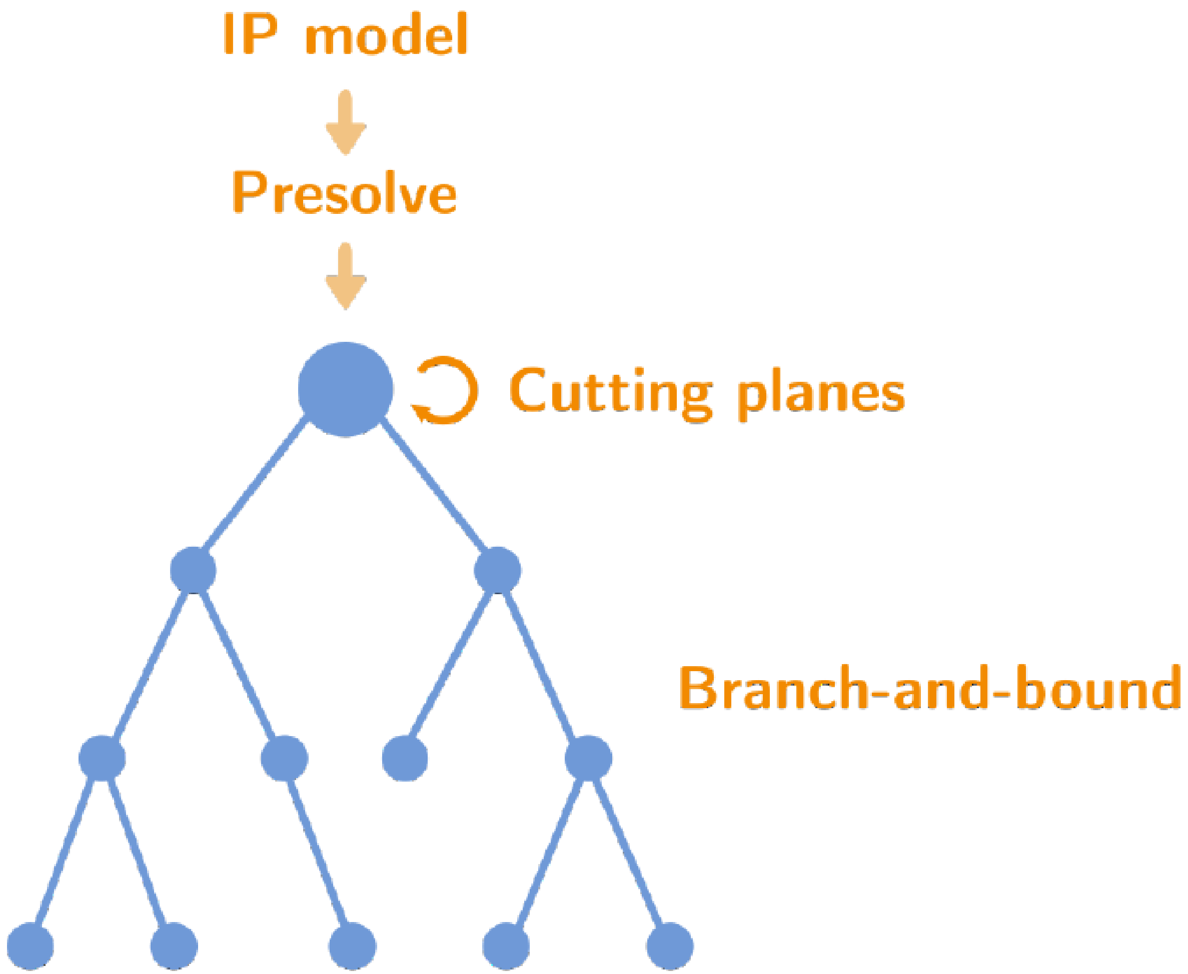


upper bound: 3

Categories of Successful Applications

- Sequencing and routing problems
  - single machine scheduling with setup times, time windows, precedence constraints (including TSPTW) [Cire & vH, OR2013], [Kinable et al. EJOR 2017] [O'Neil & Hoffman, ORL2019]
- Decomposition and embedding in MIP models
  - nonlinear objective functions [Bergman&Cire, MgtSc 2018]
  - column generation [Morrison et al. IJOC 2016] [Kowalczyk & Leus IJOC 2018]
- Combinatorial optimization
  - MISP, MAX-CUT, MAX-2SAT, … [CPAIOR 2011, 2012] [IJOC 2014, 2016] [J Heur 2014]
- Constraint Programming
  - DD-based constraint propagation [Andersen et al. CP2007] [Hoda et al. CP2010]

Application to Integer Programming



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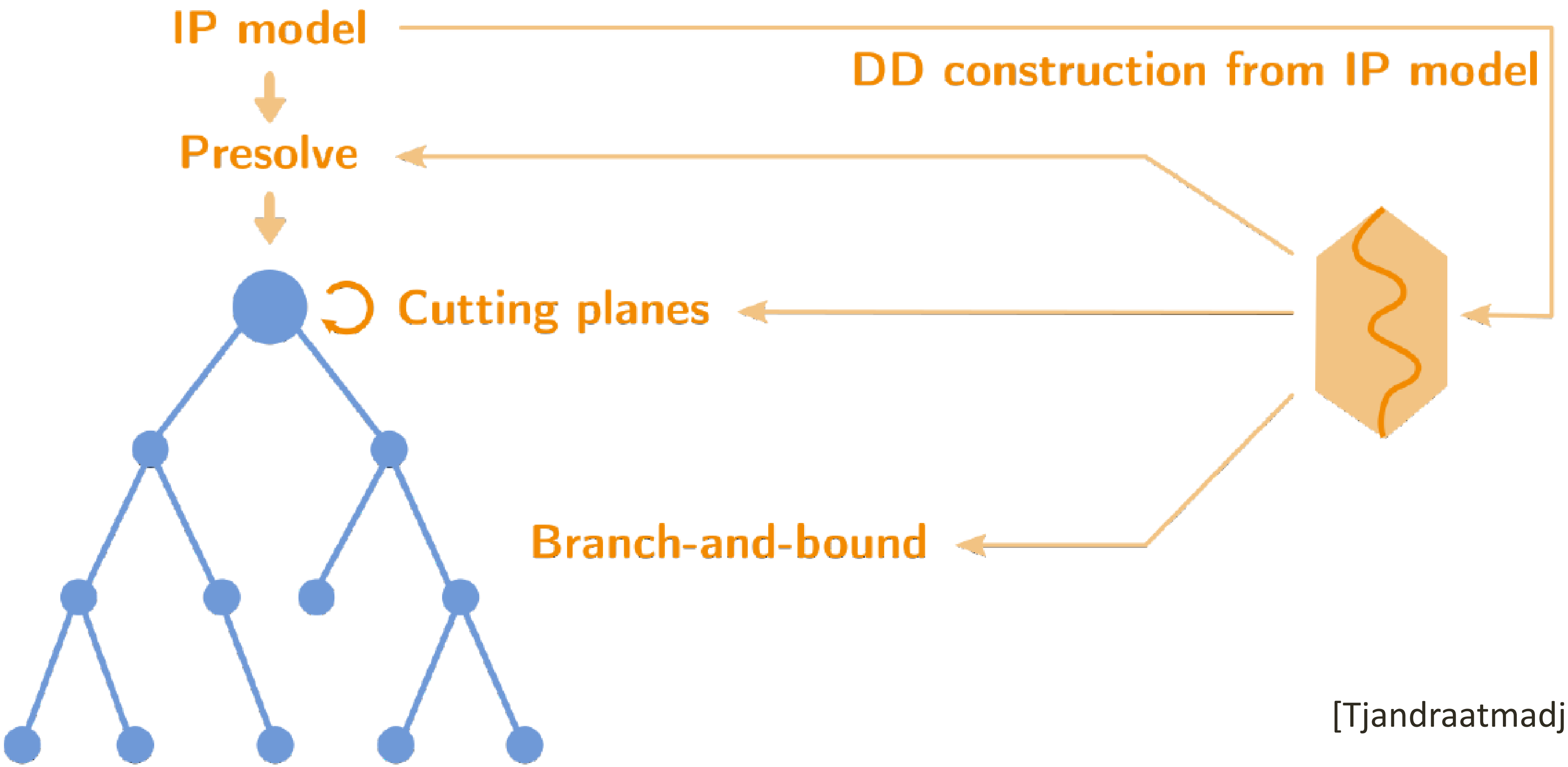

[Tjandraatmadja, PhD 2018]

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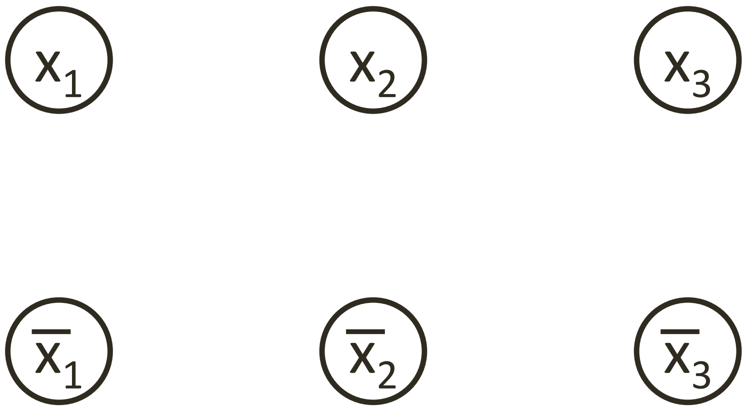
- Option 3: use structure inferred by solver
  - conflict graph/clique table

Conflict Graph for Binary Problems



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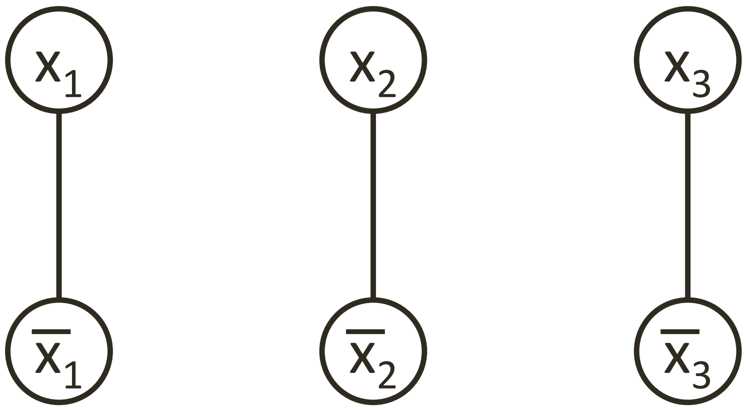

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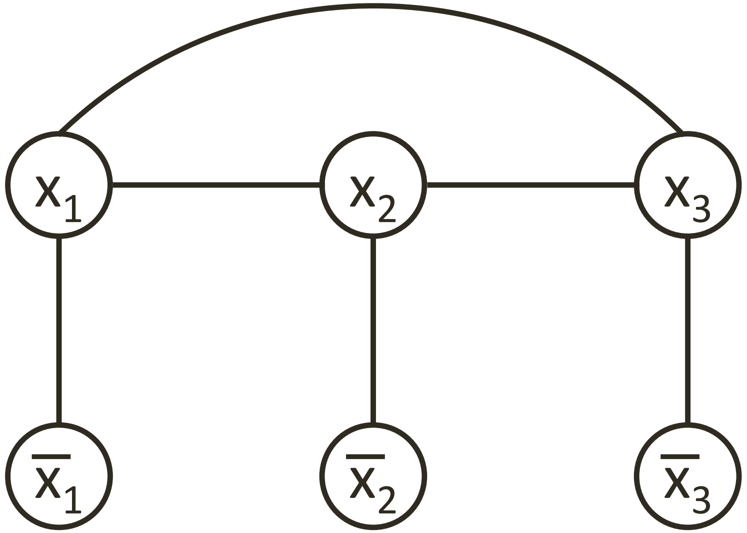

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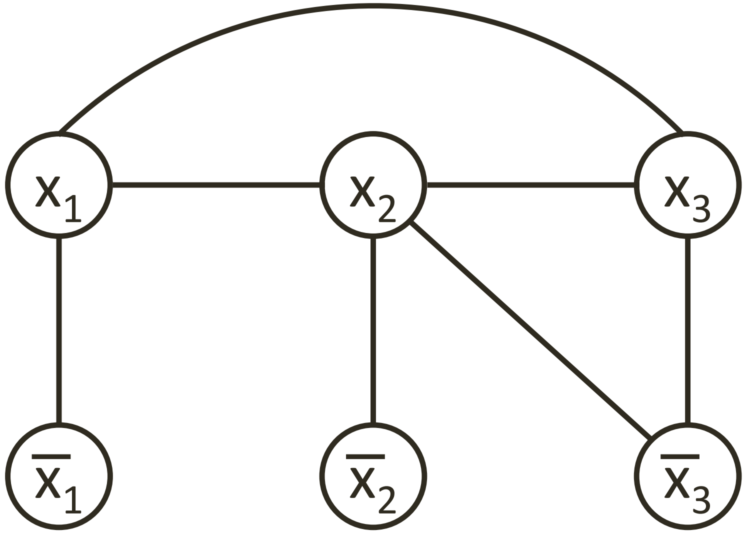

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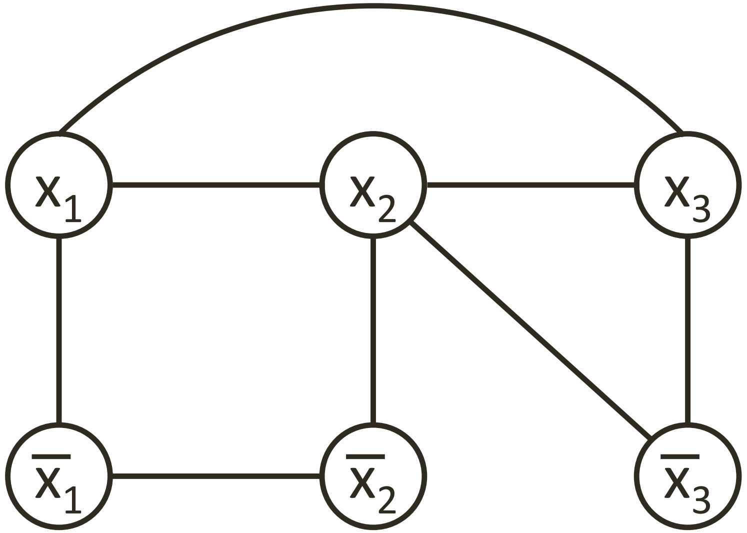

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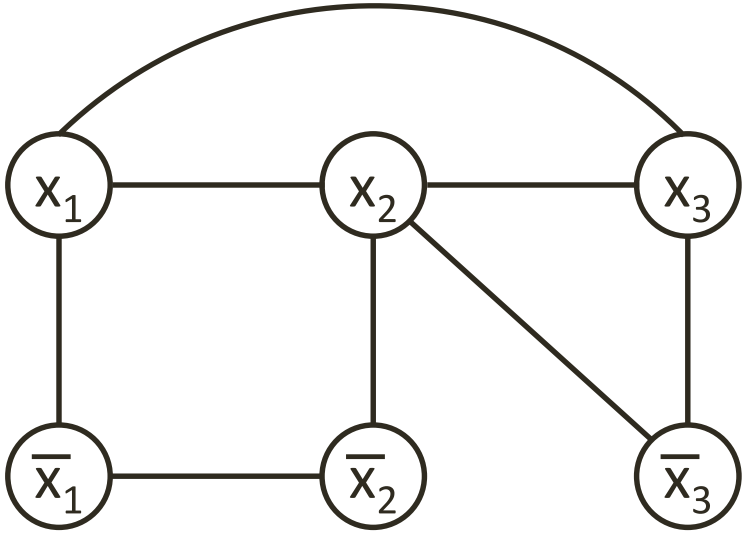

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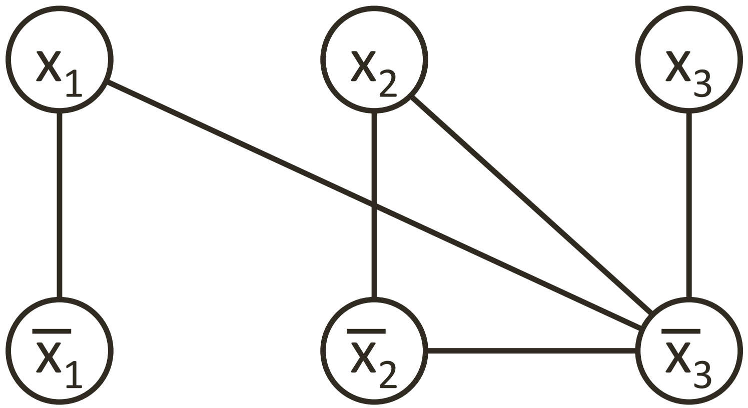
Conflict graphs are inferred and constructed by most modern MIP solvers

[Atamtürk et al., 2000; Achterberg, 2007]

Decision Diagram Compilation

- State: variable domains
- Transition: propagate decision



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x_1

x_2

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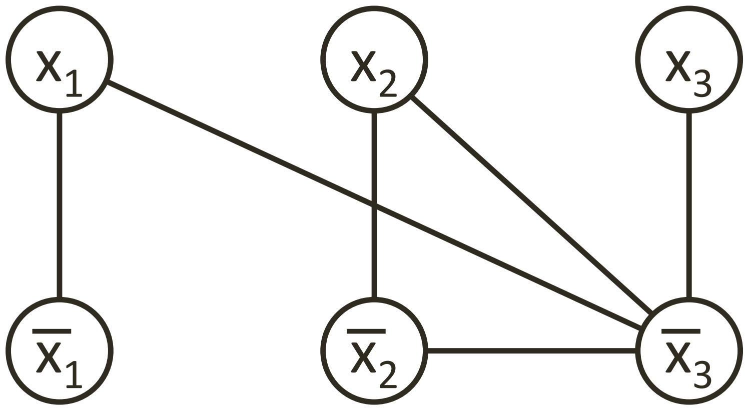

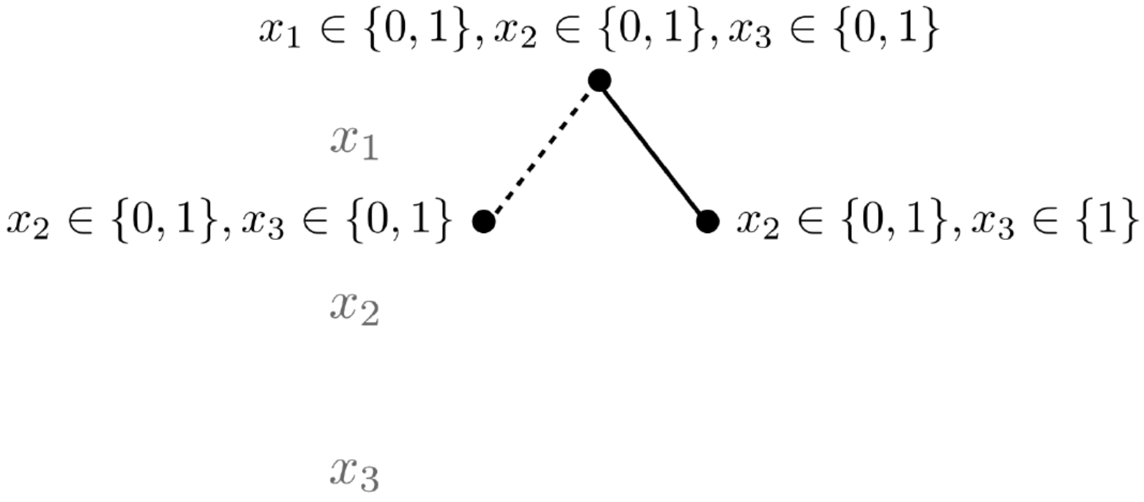

Decision Diagram Compilation

- State: variable domains
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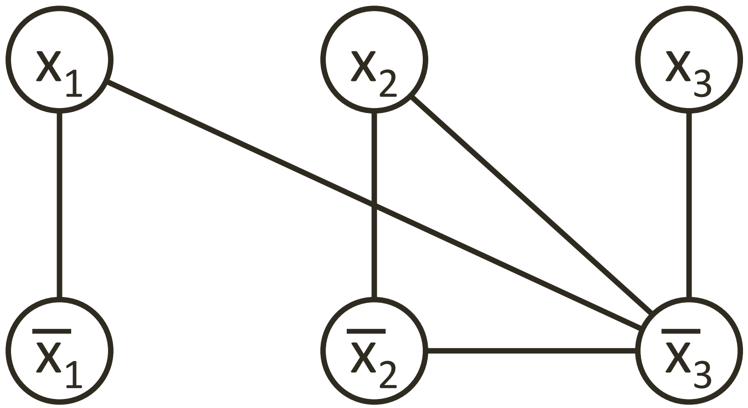

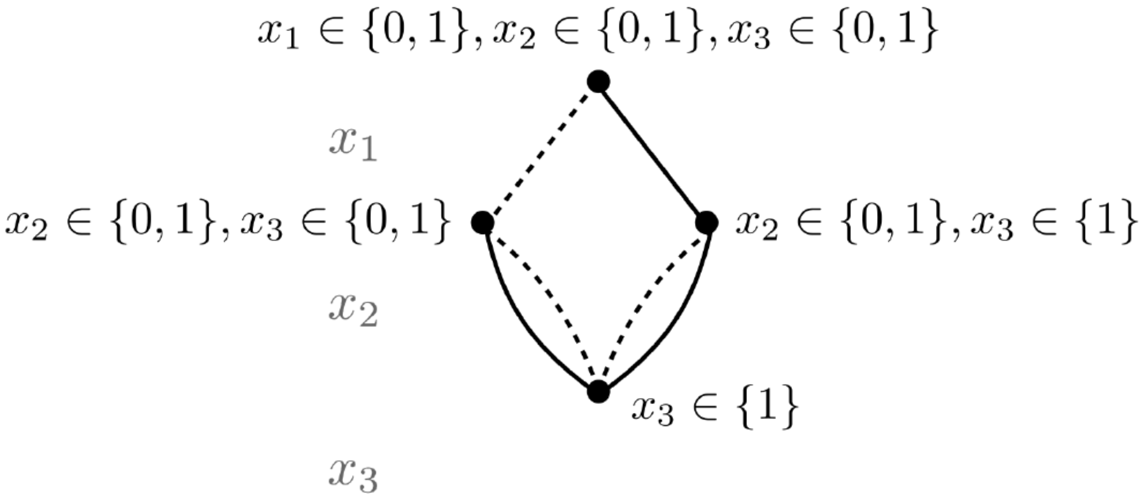

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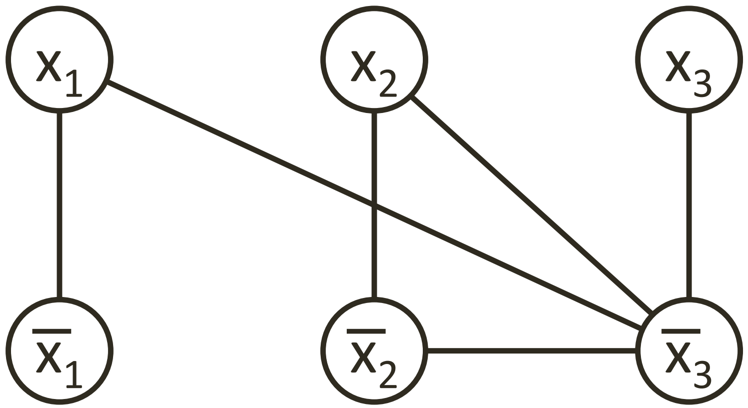

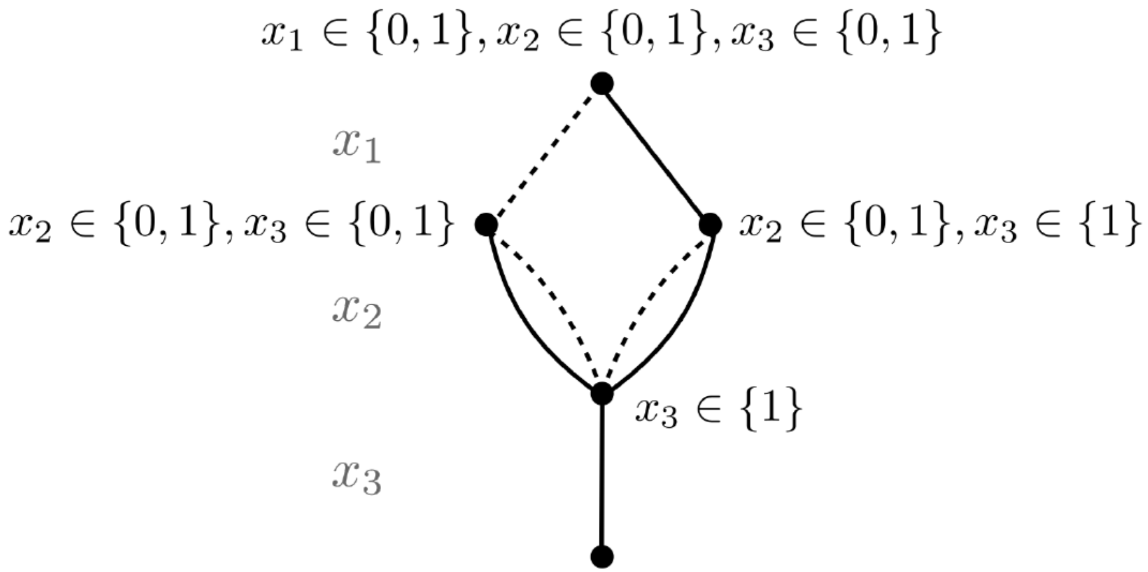

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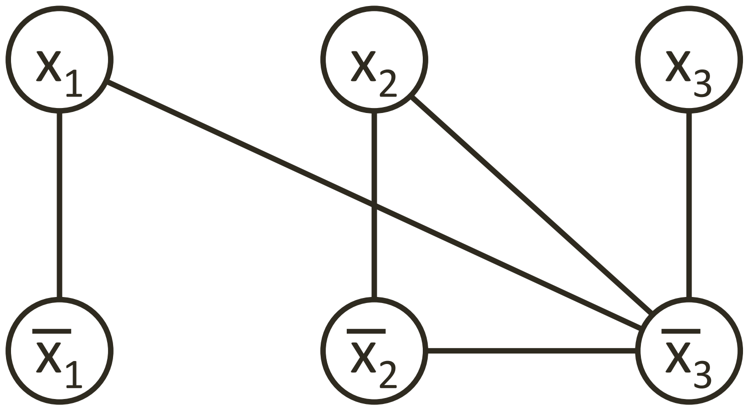

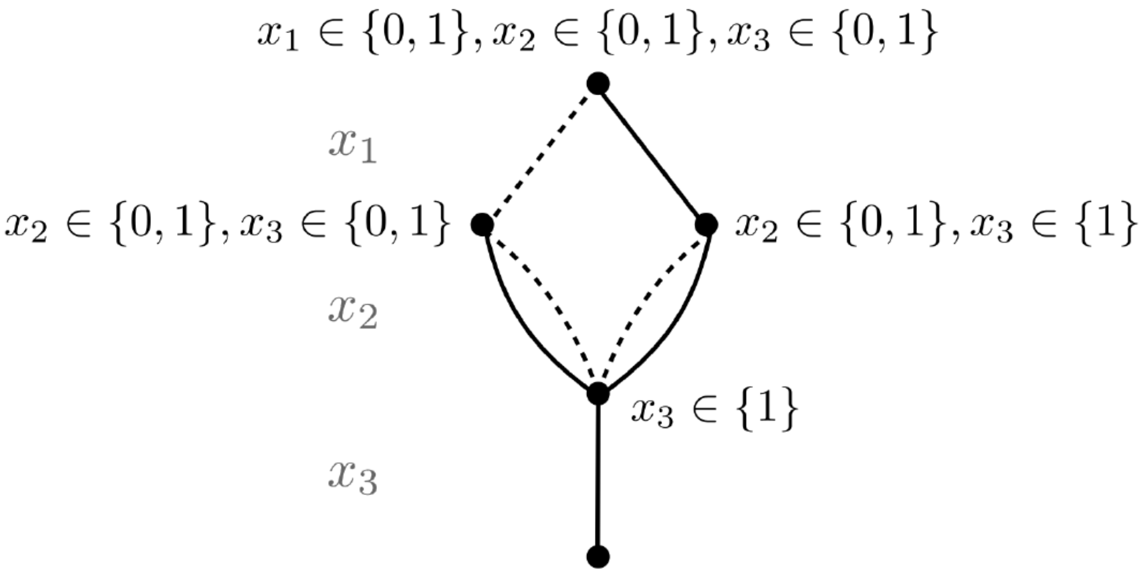

- Theorem: If root state is domain consistent, then this approach yields a reduced exact DD

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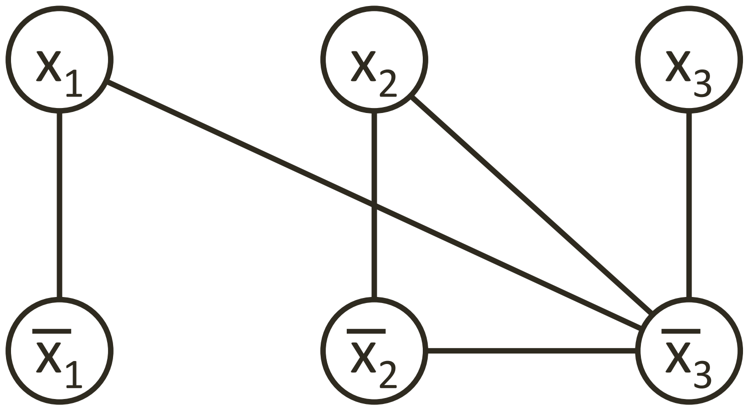

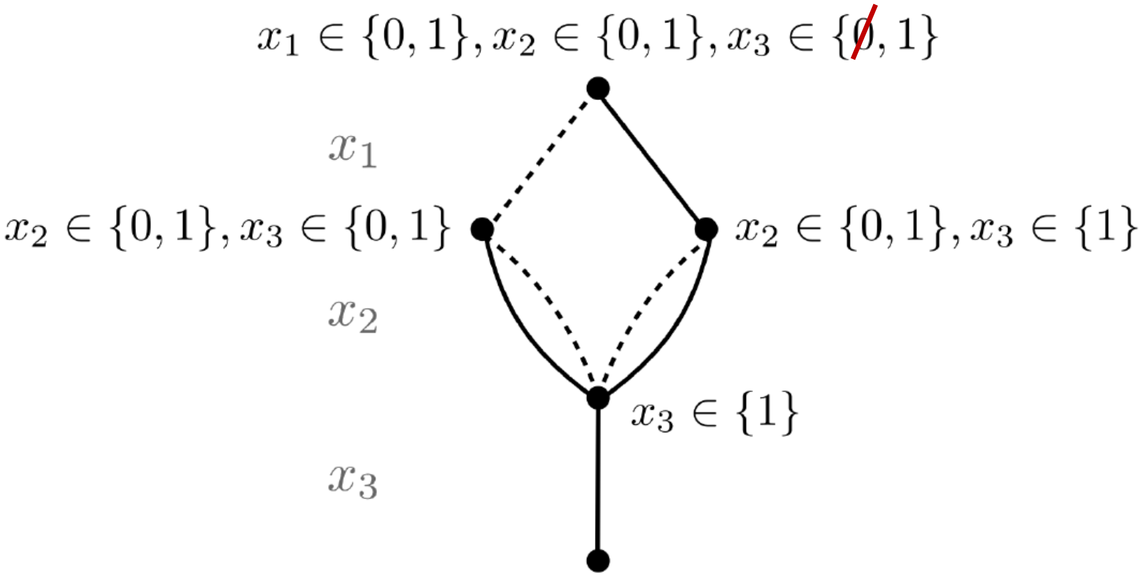

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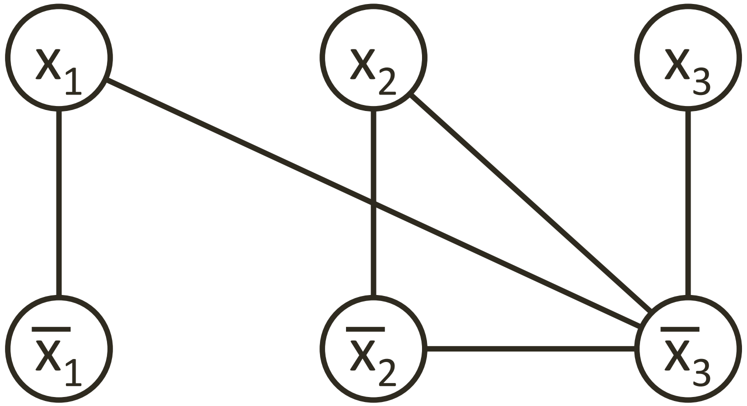

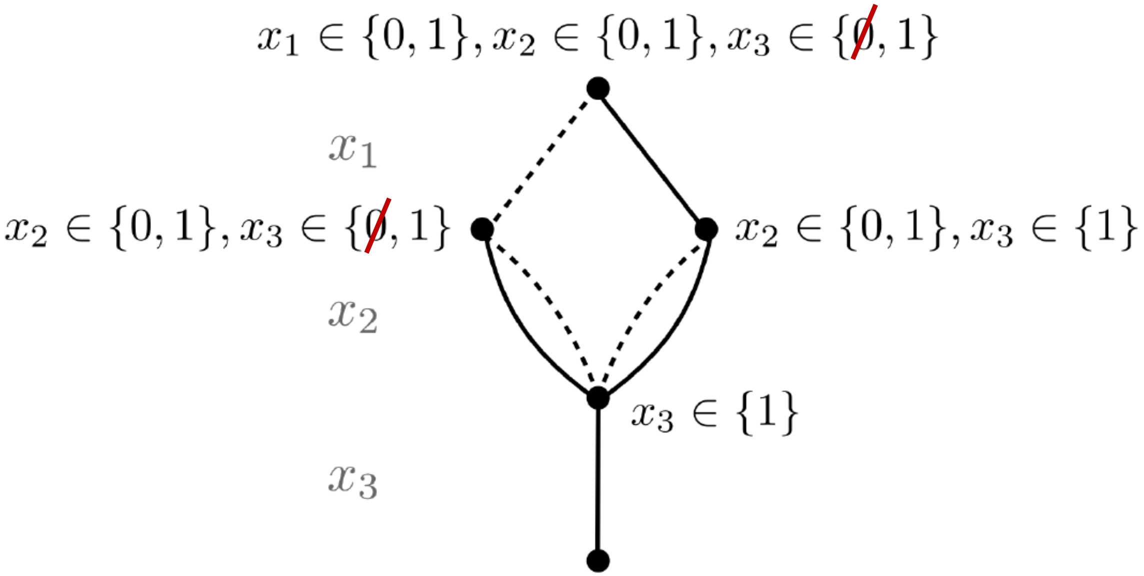

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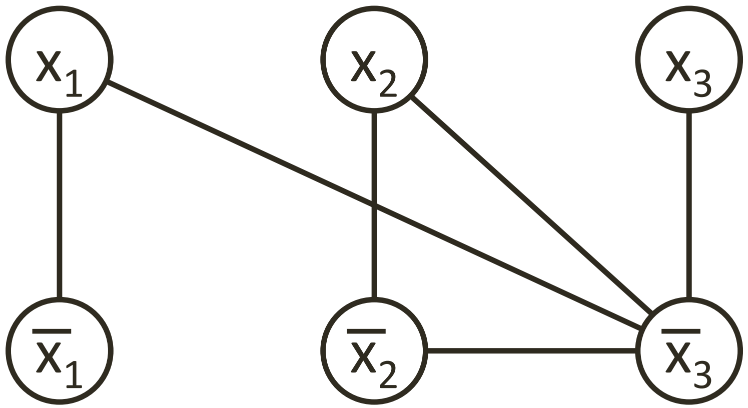

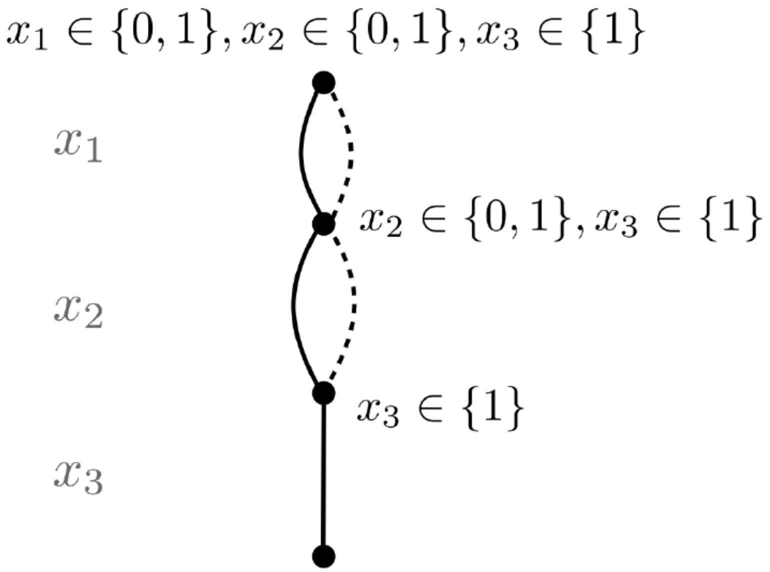

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Stronger DD relaxation via Lagrangian

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Original IP model

$$\max \; c^\top x$$

$$Fx \leq f \quad \leftarrow \text{Structured constraints for DD}$$

$$Ax \leq b \quad \leftarrow \text{Any set of linear constraints}$$

$$x \in \mathbb{Z}^n, \;\; \ell \leq x \leq u$$

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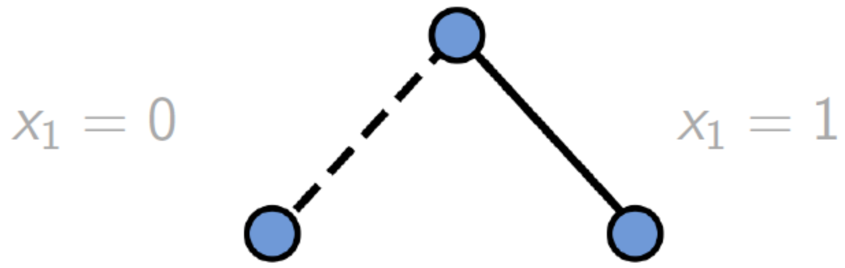
Lagrangian subproblem is longest path in DD (efficient)

Stronger DD relaxation via Propagation

- Propagate linear constraints
- Additional state information
  - – variable domains
  - – constraint right-hand sides

$$3x_1 + x_2 + 2x_3 \leq \mathbf{4}$$

$$x_1 \in \{0,1\}, x_2 \in \{0,1\}, x_3 \in \{0,1\}$$



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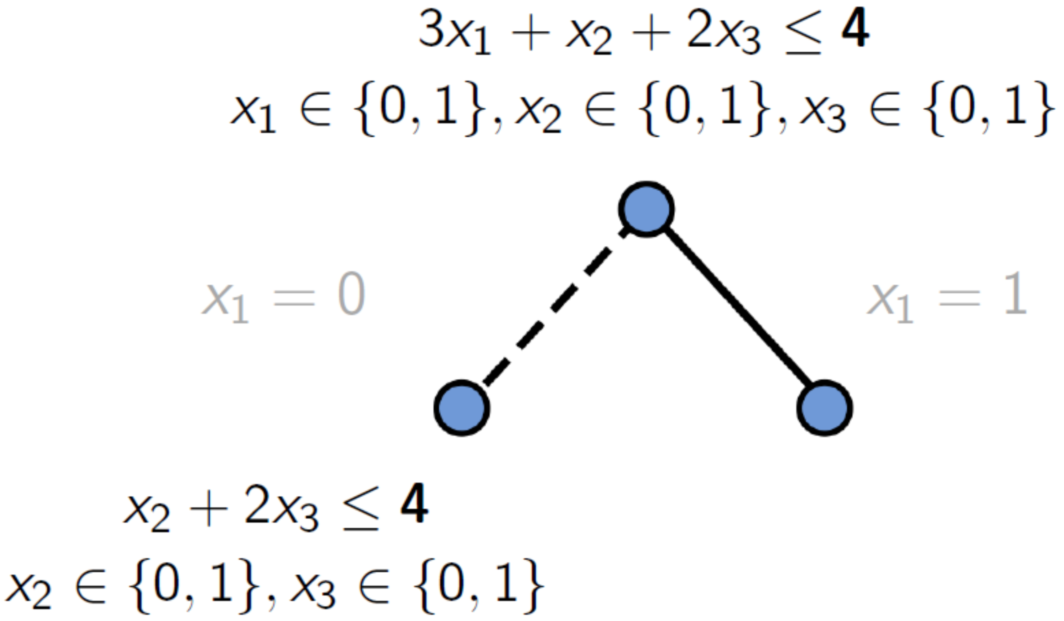

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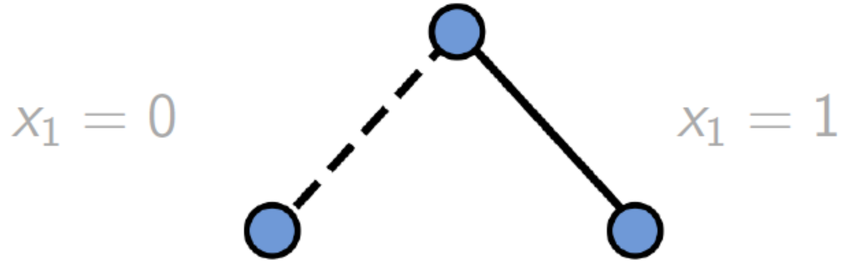
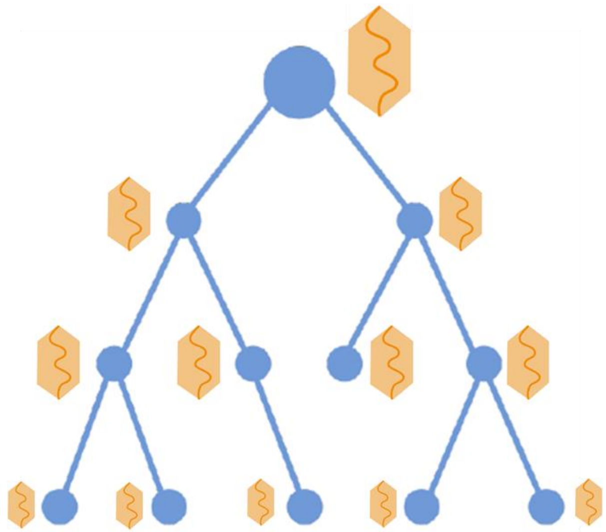


$$x_2 + 2x_3 \leq \mathbf{4} \qquad\qquad x_2 + 2x_3 \leq \mathbf{1}$$
$$x_2 \in \{0,1\}, x_3 \in \{0,1\} \qquad\qquad x_2 \in \{0,1\}, \mathbf{x_3 \in \{0\}}$$

Experimental evaluation

- Experimental setup
  - Independent set problem on random graphs (Watts-Strogatz)
  - Add set of random knapsack constraints $\sum_{i \in S} a_i x_i \leq b$
  - Vary number of variables n
  - Vary number of knapsack constraints m



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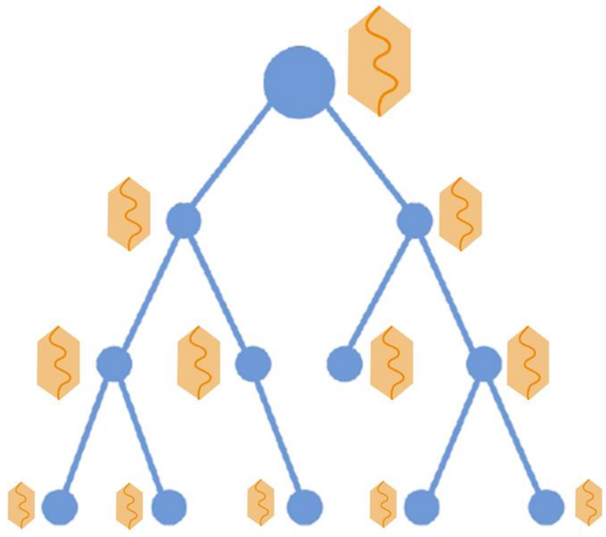
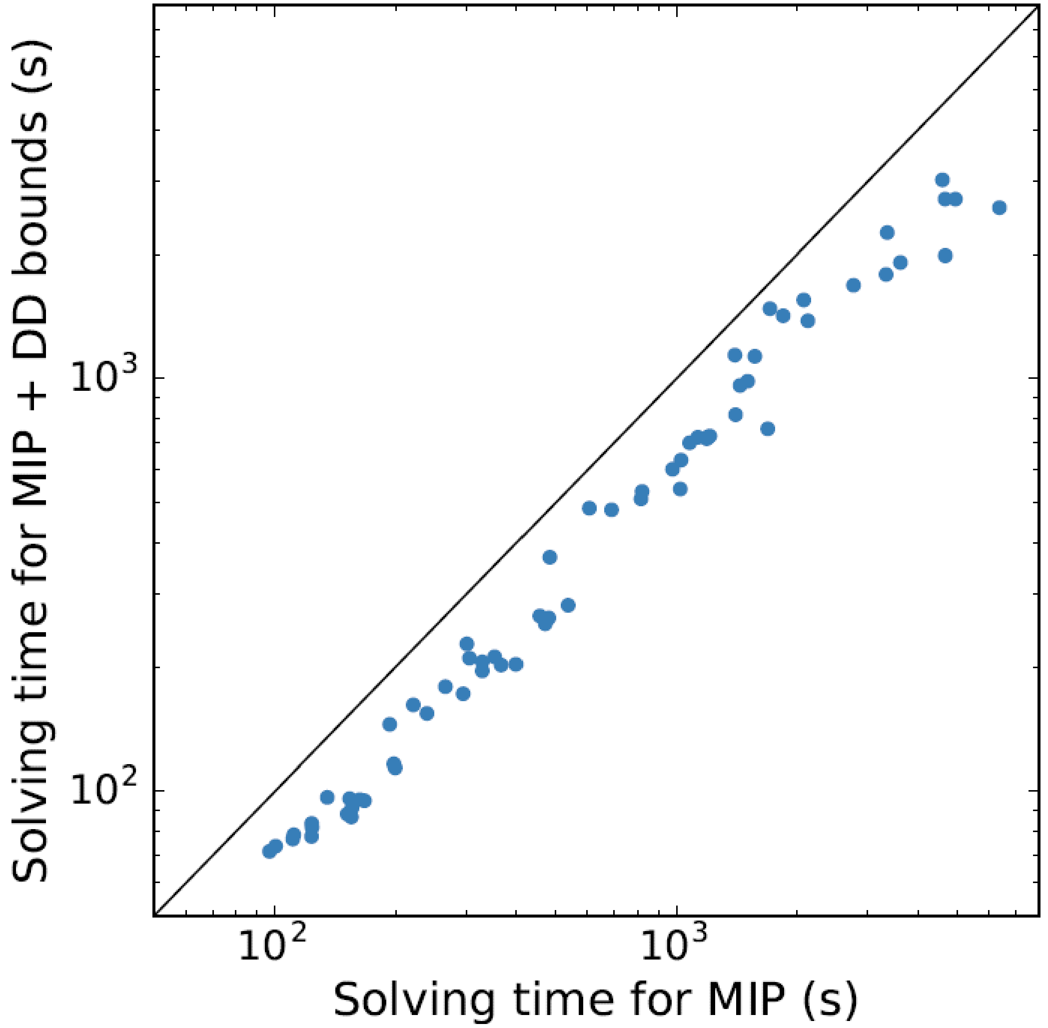
  - – Vary number of knapsack constraints m
- Implemented in SCIP 5.0.1
  - – Only IP model is given to solver
  - – DD compiled automatically



Random Graphs + Knapsack Constraints



n = 300, 350, 400, 450
m = 0.1n

On average: 65.5% node reduction
1.59x speedup

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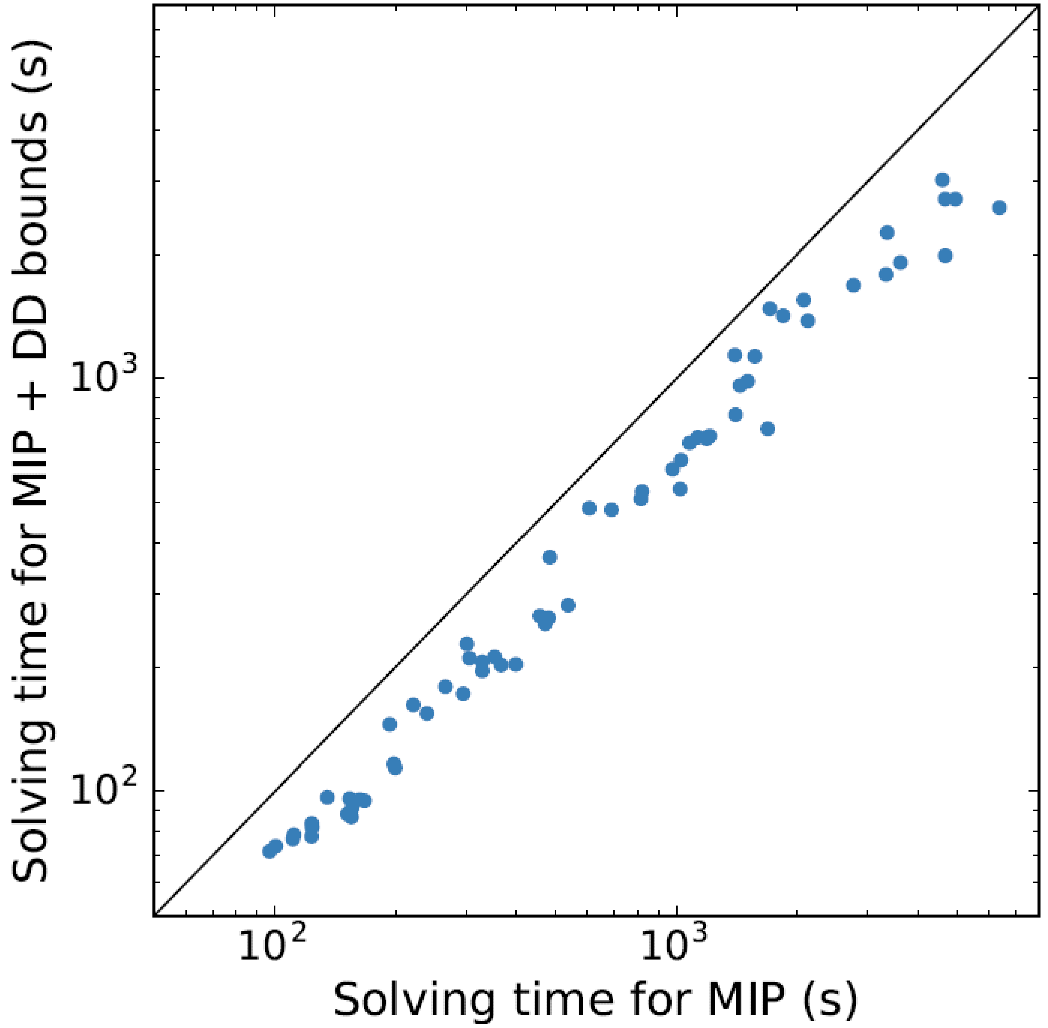

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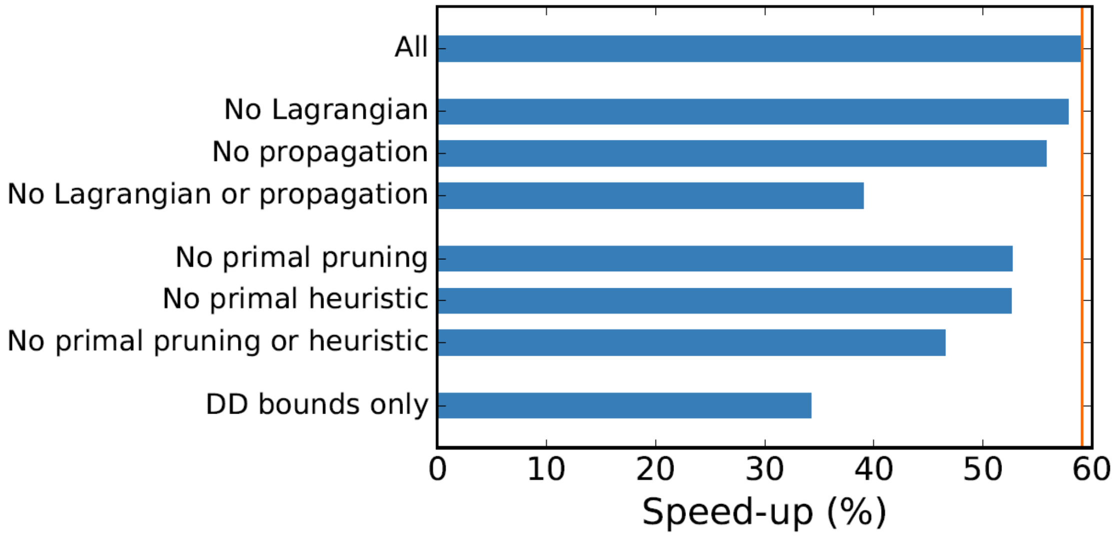
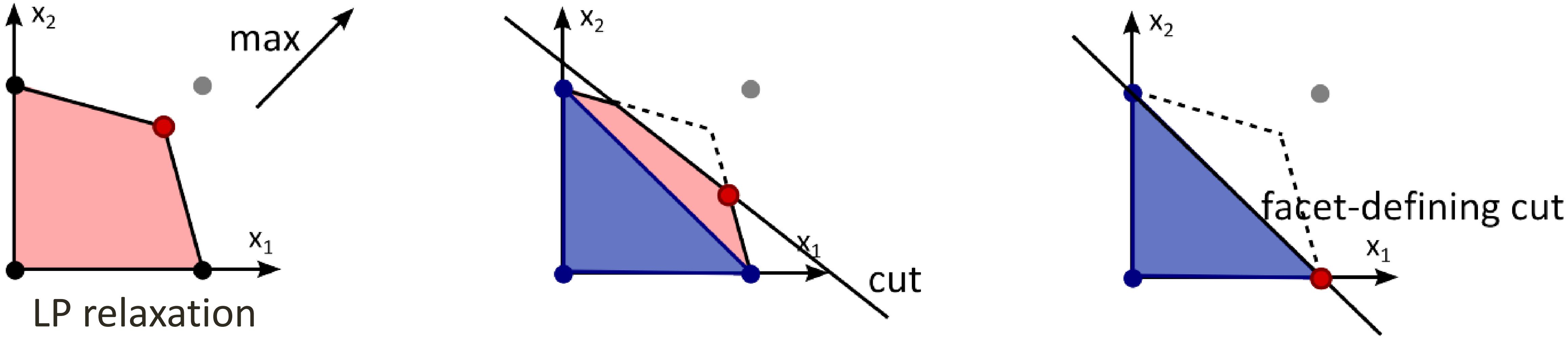


Deriving Cutting Planes from Decision Diagrams



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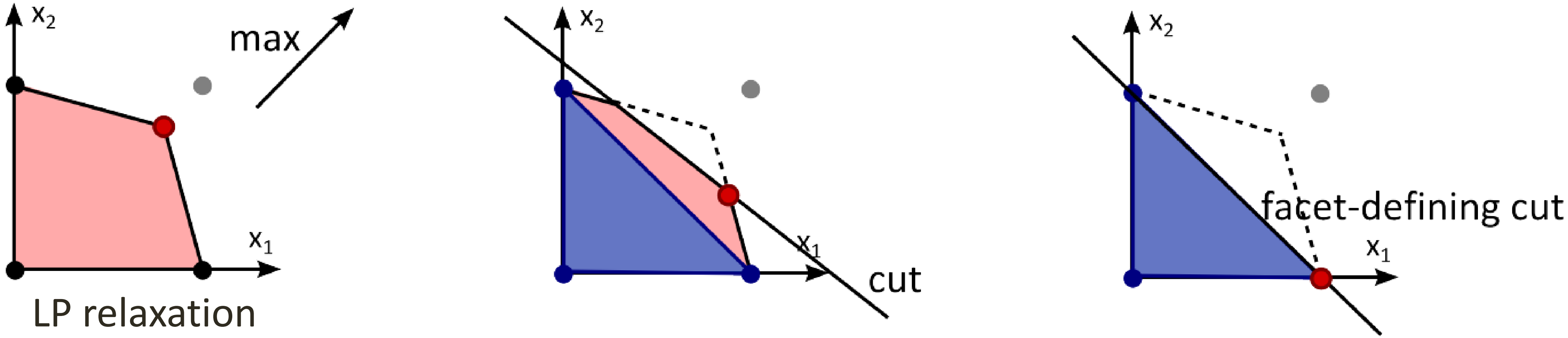
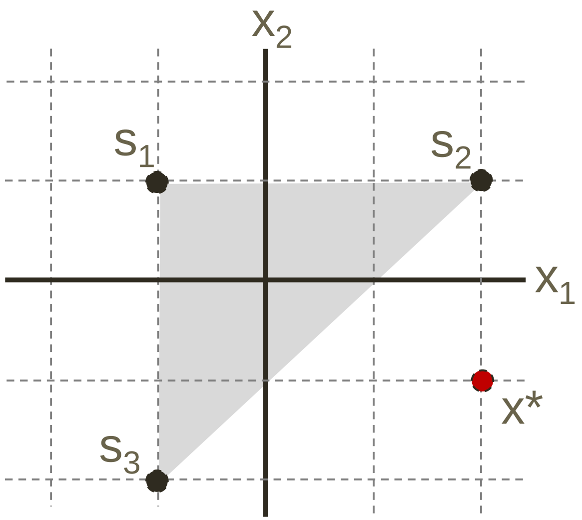


Related work:

- Becker et al. [2005], Behle [2007]: Lagrangian cut generation using exact decision diagrams
- Buchheim et al. [2008]: Target cuts

Cut-Generating Linear Program



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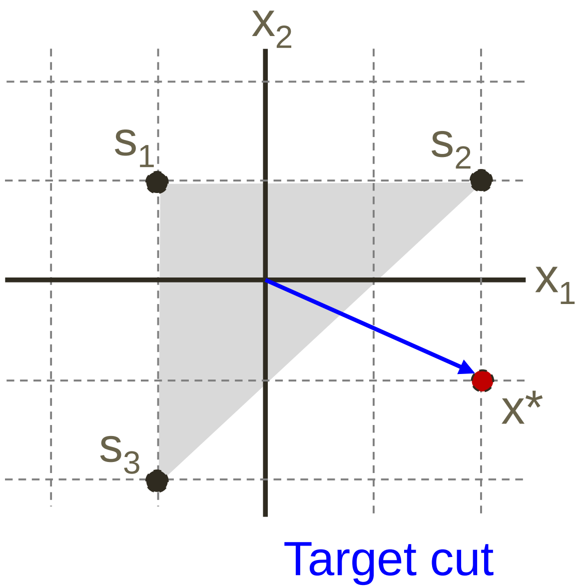

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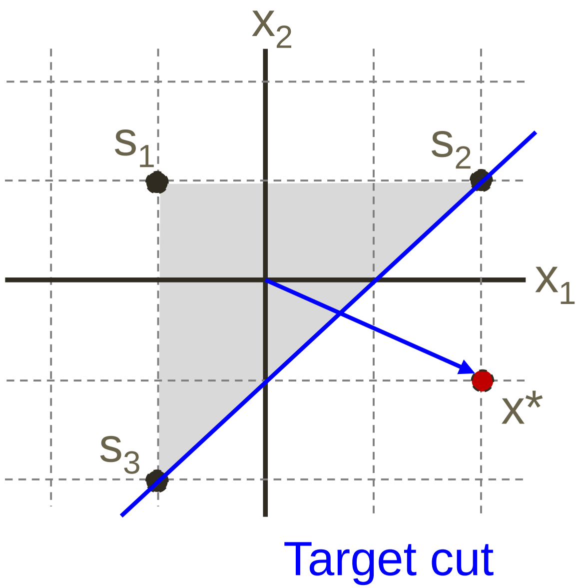

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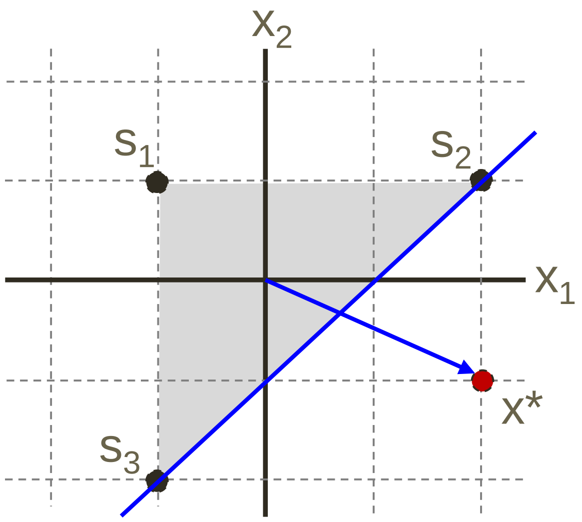
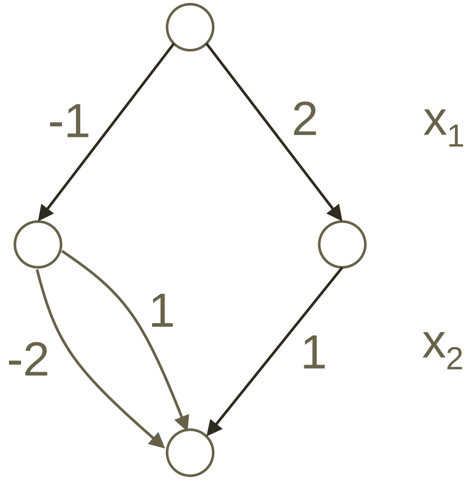


Target cut



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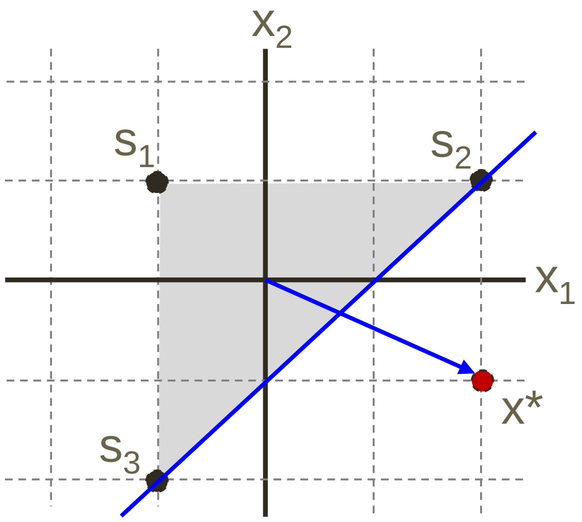

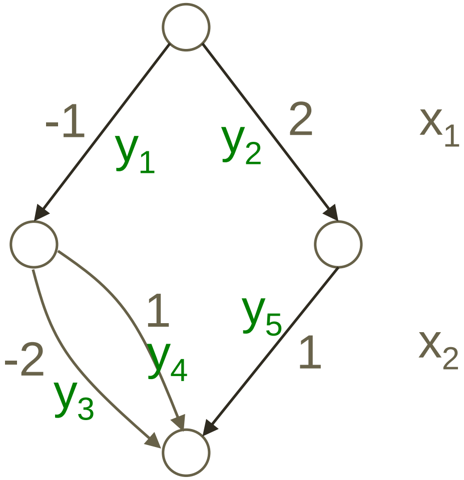

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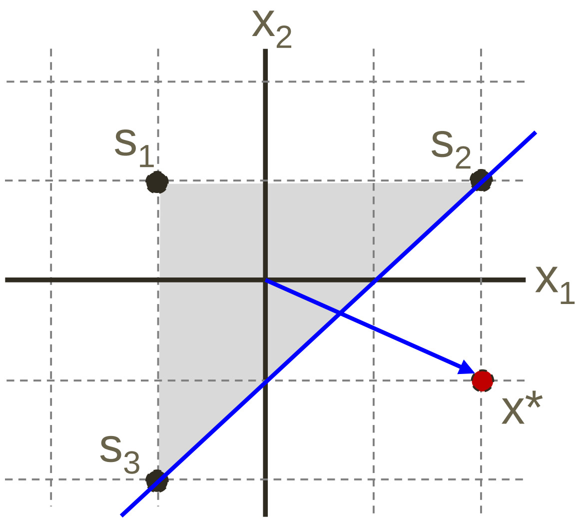
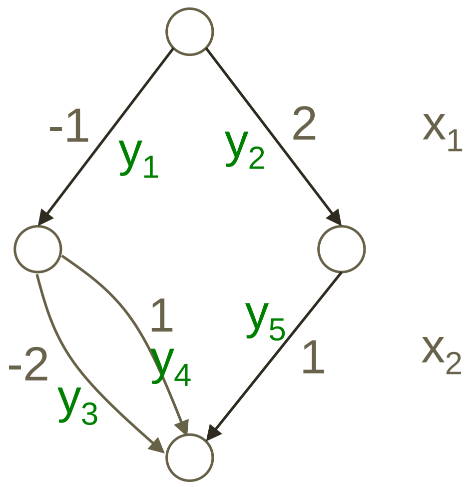




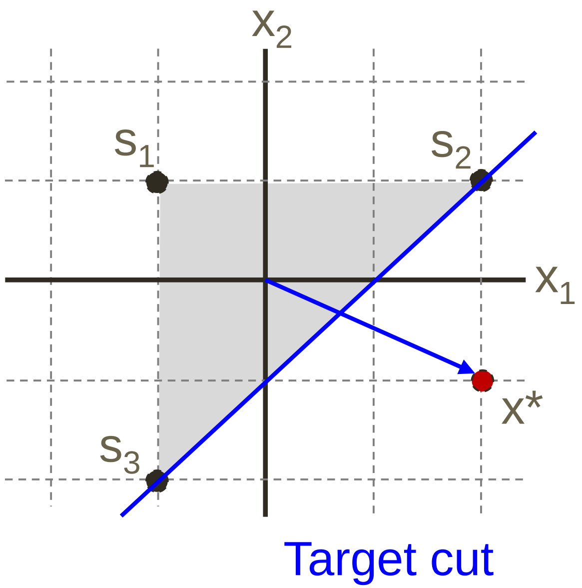
min $y_1 + y_2$

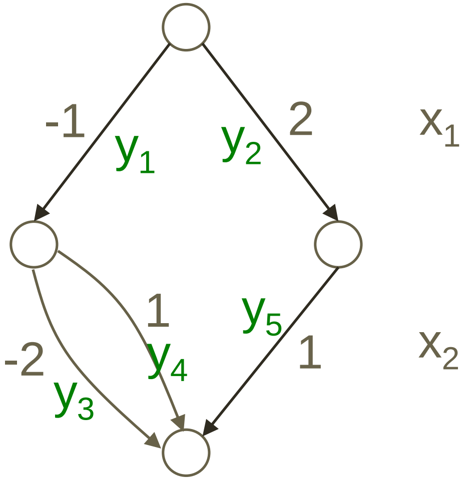
s.t. $-y_1 + 2y_2 = 2$

$-2y_3 + y_4 + y_5 = -1$

\+ flow conservation

Cut-Generating Linear Program





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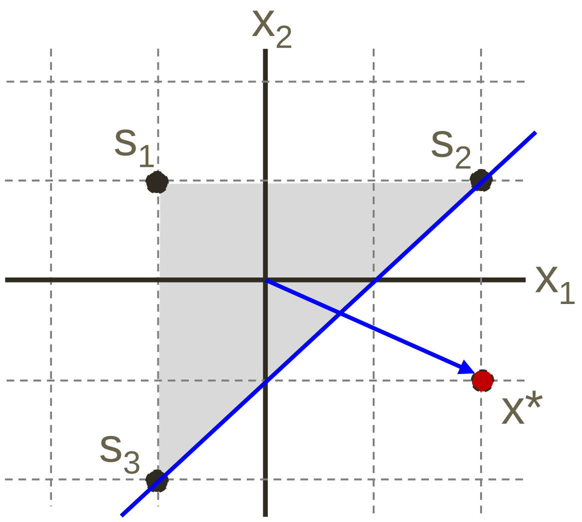
Solution:

$y_1=y_3=4/3$, $y_2=y_5=5/3$, $y_4=0$

Cut-Generating Linear Program



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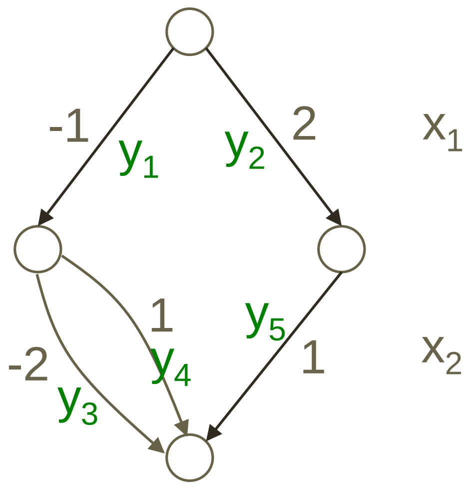

$$\begin{aligned} \min\ & y_1 + y_2 \\ \text{s.t.}\ & -y_1 + 2y_2 = 2 \\ & -2y_3 + y_4 + y_5 = -1 \\ & + \text{flow conservation} \end{aligned}$$

Solution:
$y_1 = y_3 = 4/3$, $y_2 = y_5 = 5/3$, $y_4 = 0$

- Solution methods
  - solve CGLP as LP (facet defining cuts) [Tjandraatmadja & vH, IJOC 2019]
  - or use subgradient method (iteratively finds longest path in DD) [Davarnia & vH]

Outer Approximation Scheme for MINLP

- Solve Integer Linear Programming relaxation: x*
- For all constraints that are violated by x*: add linearization cut
- Repeat until x* is feasible

- Requires that all functions are convex and sufficiently smooth (continuously differentiable)

[Duran and Grossmann, 1986]

[Westerlund & Pettersson, 1995]

Outer Approximation with DDs

- Generate a DD (relaxed or exact) for each individual constraint
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  - For all constraints that are violated by x^*: add DD cut
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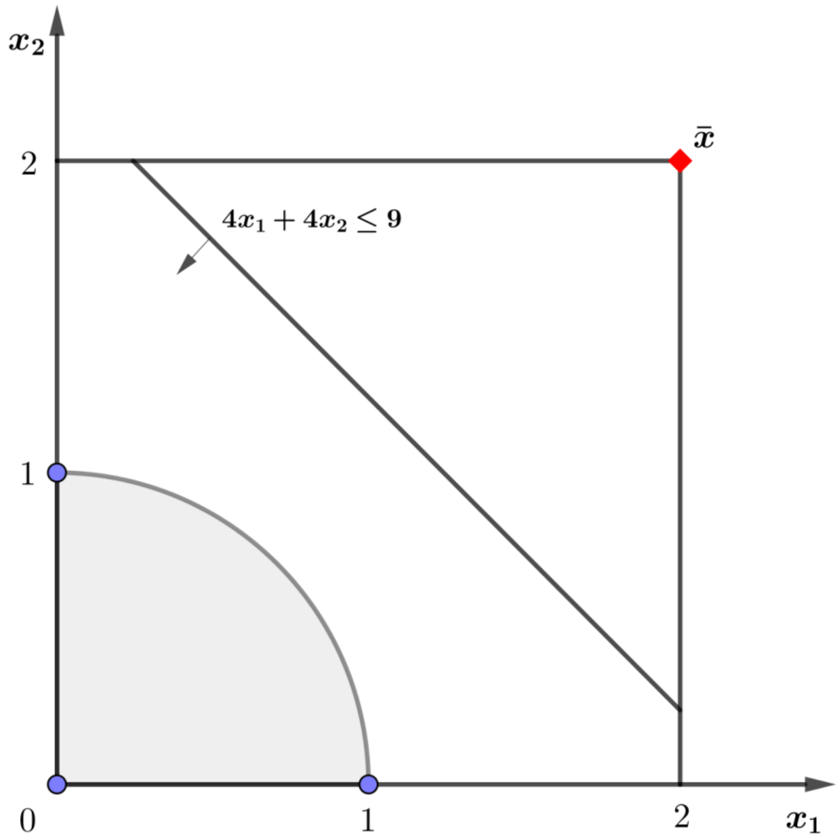
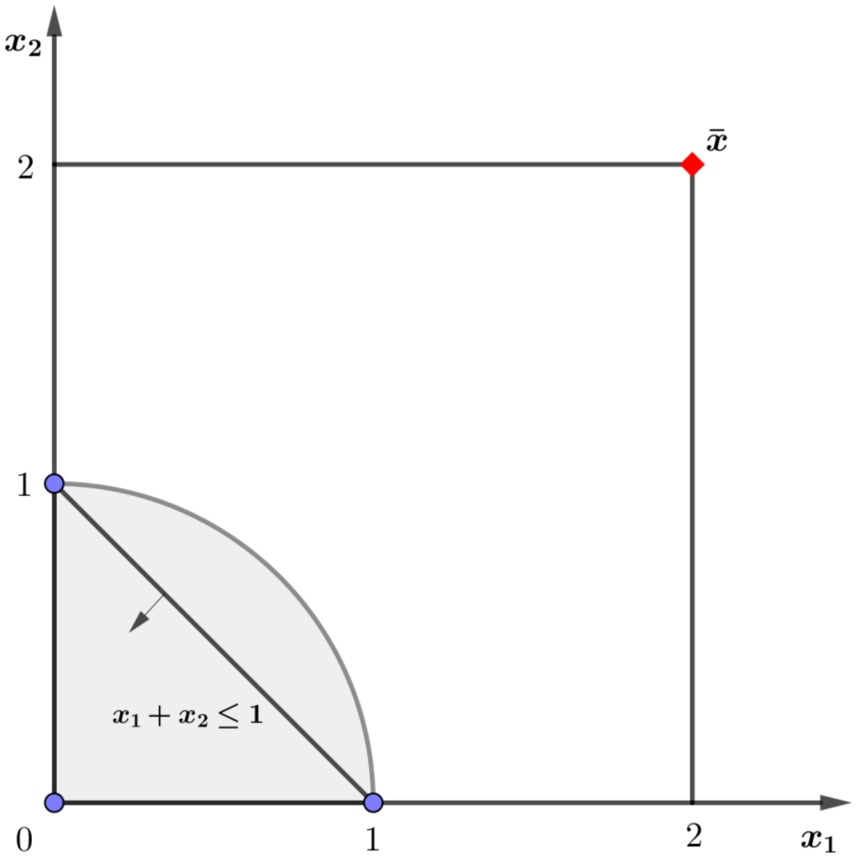
- Requires that all functions are factorable
  - Can be non-convex

Outer Approximation Example

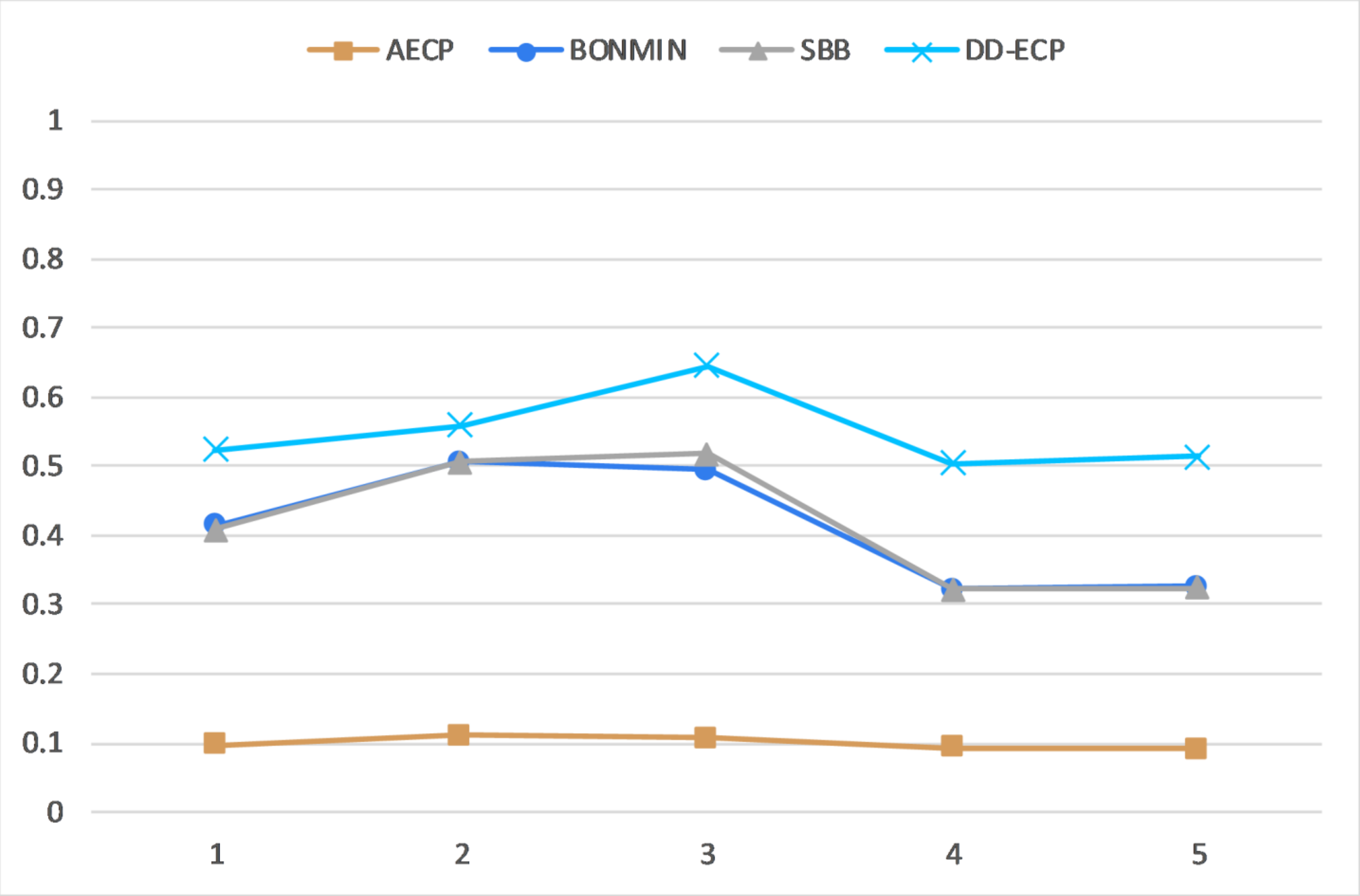


Linearization cut



DD cut

Experimental Evaluation: Polynomial Knapsack



Gap closure for various outer approximation methods

$$\max \quad \sum_{i=1}^{n} c_i x_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n} a_i^j x_i^{k_i^j} \leq b_j \quad \forall j \in J$$

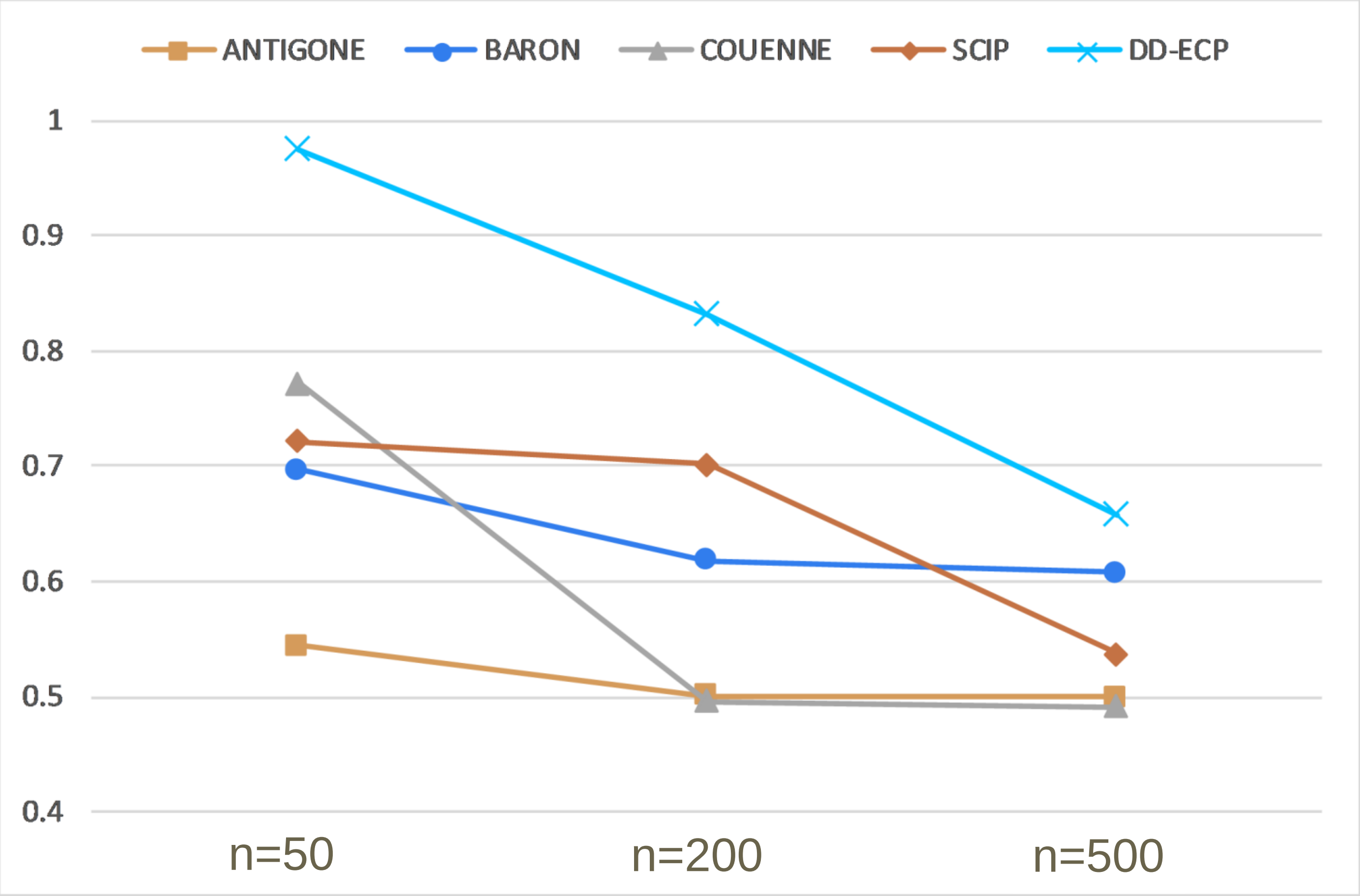
$$\boldsymbol{x} \in [\boldsymbol{l}, \boldsymbol{u}] \cap \mathbb{Z}^n$$

n=500, |J| = 5, bounds [0,5]
degree k of monomial in {1,…,10}

5 randomly generated instances

maximum DD width is 3000
time limit is 300s

Experimental Evaluation: Penetration Pricing



Gap closure for various sizes and MINLP solvers

$$\min \quad \sum_{i=1}^{n} c_i x_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n} a_i^j x_i e^{-x_i^{k_i^j}} \geq b_j, \quad \forall j \in J$$

$$\boldsymbol{x} \in [\boldsymbol{l}, \boldsymbol{u}] \cap \mathbb{Z}^n.$$

Find discrete prices for n products subject to minimum revenue constraints

|J| = 5, prices {0,0.1,…,1.0}
degree k of monomial in {1,2,3}

maximum DD width is 5000
time limit is 300s

Conclusion

- Decision Diagrams can be applied to Integer Programming
- Incorporate DD bounds in MIP search
  - conflict graph represented as DD
  - strengthened by Lagrangian relaxation and constraint propagation
  - up to 65.5% node reduction (1.59x speedup)
- Outer approximation for MINLP
  - applies to non-convex factorable functions
  - can outperform state-of-the-art approaches on certain problem classes

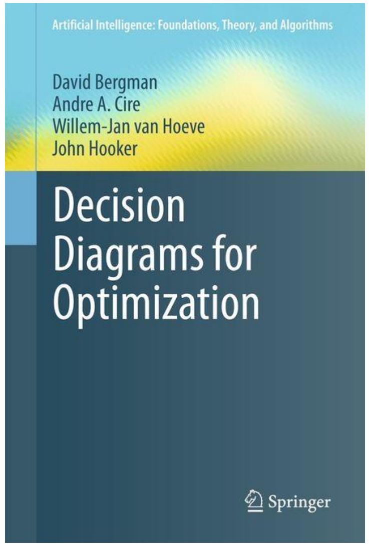
References

D. Davarnia & v.H. Outer Approximation for Integer Nonlinear Programs via Decision Diagrams. *Submitted.* (Available on *Optimization Online*.)

C. Tjandraatmadja & v.H. Incorporating Bounds from Decision Diagrams into Integer Programming. *Submitted*.

C. Tjandraatmadja & v.H. Target Cuts from Relaxed Decision Diagrams. *INFORMS Journal on Computing*, 2019.

C. Tjandraatmadja. Decision Diagram Relaxations for Integer Programming. PhD thesis, Carnegie Mellon University, 2018.



http://www.andrew.cmu.edu/user/vanhoeve/mdd/